COMPARATIVE EXPECTATIONS

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ABSTRACT. I introduce a mathematical account of expectation based on a *qualitative criterion of coherence* for *qualitative comparisons* between gambles (or random quantities). The qualitative comparisons may be interpreted as an agent's comparative preference judgments over options or more directly as an agent's comparative expectation judgments over random quantities. The criterion of coherence is reminiscent of de Finetti's quantitative criterion of coherence for betting, yet it does not impose an Archimedean condition on an agent's comparative judgments, it does not require the binary relation reflecting an agent's comparative judgments to be *reflexive, complete* or even *transitive*, and it applies to an absolutely *arbitrary* collection of gambles, free of structural conditions (e.g., boundedness, measurability, closure under algebraic or topological operations, etc.). Moreover, unlike de Finetti's criterion of coherence, the qualitative criterion respects the *principle of weak dominance*, a standard of rational decision making that obliges an agent to reject a gamble that is possibly worse and certainly no better than another gamble available for choice. Despite these weak assumptions, I establish a qualitative analogue of de Finetti's Fundamental Theorem of Prevision, from which it follows that any coherent system of *comparative expectations* can be extended to a weakly ordered coherent system of comparative expectations over any collection of gambles containing the initial set of gambles of interest. The extended weakly ordered coherent system of comparative expectations satisfies familiar additivity and scale invariance postulates (i.e., independence) when the extended collection forms a linear space. In the course of these developments, I recast de Finetti's quantitative account of coherent prevision in the qualitative framework adopted in this article. I show that *comparative previsions* satisfy qualitative analogues of de Finetti's famous bookmaking theorem and his Fundamental Theorem of Prevision.

The results of this article complement those of another article [Pedersen, 2013]. I explain how those results entail that any coherent weakly ordered system of comparative expectations over a unital linear space can be represented by an *expectation function* taking values in a (possibly non-Archimedean) totally ordered field extension of the system of real numbers. The ordered field extension consists of *formal power series* in a *single* infinitesimal, a natural and economical representation that provides a relief map tracing numerical non-Archimedean features to qualitative non-Archimedean features.

INTRODUCTION

An influential view in the foundations of probability portrays an intimate connection between, on the one hand, rational credences and expectations and, on the other hand, rational decisions made under conditions of uncertainty. On this view, criteria of rational decision

I would like to extend my gratitude to Kenny Easwaran, Frederik Herzberg, Konstantinos Katsikopoulos, Marcus Pivato, Alex Pruss, Dana Scott, Stanislav Speranski, Peter Wakker, and Greg Wheeler for valuable comments and discussion. I would also like to thank the anonymous reviewers for their helpful remarks. This paper is dedicated to the memory of Horacio Arló-Costa.

making entail prescriptive laws which regulate rational credences and, more generally, rational expectations. The work of Ramsey [1926], Savage [1954], and Anscombe and Aumann [1963] are classic examples of formal developments that adopt this decision-theoretic perspective in bids to secure firm foundations for normative theories of belief and decision that numerically represent rational value judgments, credences, and expectations in models of subjective expected utility.

De Finetti's bookmaking approach is another example proceeding from this decision-theoretic perspective. De Finetti [1931, 1937, 1974a,b] famously advanced a single criterion of coherence according to which an agent should not endorse preferences which can be linearly combined to result in a uniform sure loss in gambling. De Finetti's approach to the foundations of probability, like other prominent approaches pursued from a decision-theoretic point of view, has been the subject of generous praise and ample criticism, some pieces of which apply generally to decision-theoretic approaches, others specifically to de Finetti's approach. In this article I shall introduce a theory of expectation that, among other things, incorporates commendable features of de Finetti's account and reckons with an important line of criticism which applies generally to standard decision-theoretic approaches to the foundations of probability, but which made its unequivocal debut as a forceful objection to de Finetti's bookmaking approach.

De Finetti [1931, 1937] first adopted his bookmaking approach to expound his subjectivistic conception of probability around the time that Ramsey [1926] independently developed his ideas on probability.[1] Later [1974a, 1974b] he adapted this approach to explicating his subjectivistic conception of expectation—called *prevision* by de Finetti—which subsumes his conception of probability as a special case.

Fundamental to de Finetti's subjectivistic account is the assumption that each agent at any particular given time is in a state of information with respect to which he distributes his previsions (expectations) over a given collection of random quantities of interest to him at that moment. The state of information of a given agent at a given time comprises the stock of certainties (background knowledge, settled assumptions, evidence, etc.) to which he is committed at that time, serving as a standard for determining whether he is committed at that moment to judging a given proposition to be possible or impossible according to whether or not it is logico-mathematically consistent with his stock of certainties—and, correspondingly, for determining whether he is committed at that moment to judging a given proposition to be certain according to whether he is committed to judging its negation to be impossible. In de Finetti's account it is presumed that each agent at any given time is committed to being certain that each random quantity of interest at that moment assumes a unique real value from among one or more real values which he is committed to judging as possible for the quantity on the basis of his current state of information. Thus, a random quantity is a well-determined real number whose true value may be unknown to a given agent based on his state of information at a given time. In de Finetti's treatment, an event is a proposition identified with a random quantity admitting at most two numerical values, one and zero, corresponding to true and false, respectively.

[1] De Finetti introduced two other well-known approaches during the course of his career: One based on postulates regulating qualitative probability judgments [1931, 1937], and another based on forecasting with respect to squared-error loss [1974a, 1974b].

In the present analytic formulation of de Finetti's account, every random quantity of interest to a given agent at a given time is represented by a real-valued function defined on a common set of states corresponding to a distinguished (albeit provisional) family of propositions the agent is committed to judging as individually possible (consistent), pairwise impossible (mutually exclusive), and jointly certain (collectively exhaustive) relative to his state of information at that time. In light of the intended interpretation, any real-valued function on the common set of states for a given agent at a given time is called a *random quantity*. De Finetti presumed that each agent at any given time is committed to a network of prevision judgments (expectations) about the numerical values of all random quantities of interest to him at that time and that the agent's network of prevision judgments is represented by a function that associates each random quantity of interest with a unique (possibly extended) real number expressing the agent's *prevision* about the value of the random quantity relative to his state of information at the given time. As a special case, the agent's prevision about the value of an event, a proposition represented by (the indicator function of) a subset of the given set of states for the agent, is understood to quantify his attitude of subjective probability judgment, or credence, toward the occurrence of the event. The *prevision function* so induced is accordingly interpreted to represent the agent's network of prevision judgments in terms of numerical discriminations afforded by the extended real number system.

Making mathematically precise an idea tracing as far back as Pascal and Fermat's seventeenth century correspondence on the fair division problem, de Finetti explicated the meaning of prevision, and thus subjective probability, in terms of the concept of a system of fair prices for numerical gambles [1974a, pp. 69ff.]. A gamble for a given agent at a given time is a prospect whose possible outcomes are his potential returns in amounts of a single currency denominated in units of money, gold, or some other personally valued commodity, perhaps utiles themselves. Mathematically, a (*bounded*) *gamble* is a (bounded) real-valued function on the common set of states over which the collection of random quantities of interest to the agent at the given time is defined. Given a collection of bounded gambles under consideration by a given agent at a given time, a *system of fair prices* is defined to be a functional relation—and so henceforth alternatively called a *price function* for brevity—specifying for each bounded gamble under consideration the *fair price* to which the agent is committed at the given time as his appraised value for both buying and selling the gamble. The bookmaking approach presupposes that an agent's adoption of a given price function entails that for any potential exchange of some finite number of appraised gambles at their respective fair prices, the agent is committed to judging that making the exchange is an acceptable course of action given the decision either to make the exchange or to abstain from the exchange. In order to link previsions to fair prices, the bookmaking approach further presupposes that by subscribing to a given prevision function, the agent has undertaken a commitment to adopting the prevision function as a price function where all random quantities of interest to the agent at the given time are interpreted as gambles.

De Finetti supplemented the foregoing decision-theoretic scheme for revealing an agent's system of fair prices with his criterion of coherence for gambling. According to his account, an agent's price function is *coherent* if no exchange of finitely many appraised gambles at their fair prices leads to a uniform sure net loss. The criterion of coherence requires that for every potential exchange of some finite number of appraised gambles at their fair prices and every potential decision in which abstaining from making such an exchange is an available option, the agent should not be committed to judging that making the given exchange is an acceptable course of action if, come what may, this option leads to a uniformly worse net outcome than his alternative option to abstain. Accordingly, the criterion forbids the agent from failing to be committed to rejecting (i.e., judging unacceptable for choice) the option to exchange finitely many appraised gambles although this option is *uniformly simply dominated* by the alternative option to abstain from such an exchange. An agent's price function is said to be *incoherent* if it fails to be coherent. When an agent has engaged in finitely many exchanges that lead to a uniform sure net loss—and thus is incoherent—the agent is said to be in *Dutch Book*.

Among his many significant contributions to areas such as probability, statistics, and decision theory, de Finetti established two important results based on his criterion of coherence for gambling [1949, 1974a, 1974b]. First, de Finetti showed that any price function respecting the criterion of coherence conforms to the laws of a finitely additive expectation function. In other words, he showed that any coherent price function satisfies the mathematical properties of a normalized, positive linear functional, wherever the price function is defined.

Second, de Finetti showed that any coherent price function can be extended to a coherent price function also defined for some additional bounded gamble of interest, specifying the upper and lower bounds within which the fair price of the additional gamble must take its value, and outside of which the fair price of the additional gamble would have to take its value if the extended price function were incoherent. Based on this result, called the *Fundamental Theorem of Prevision*, de Finetti proved that any coherent price function can be extended to a coherent price function also assigning a fair price to each gamble from some additional collection of bounded gambles of interest.

De Finetti's two results together entail that an agent's price function is coherent just in case it agrees with a finitely additive mathematical expectation defined on the linear span of the collection of gambles under consideration by the agent. *A fortiori*, if the collection of appraised gambles is a set of (indicator functions for) events, these results entail de Finetti's celebrated *Dutch Book Theorem*, according to which an agent's price function is coherent just in case there is a finitely additive probability function such that the fair price of each (indicator function for an) event from the given collection agrees with the probability assigned to the corresponding event. Hence, based on the criterion of coherence, it is argued that an agent's numerical credences, understood as fair prices, must conform to the mathematical laws of probability. The implications of these two results, however, extend well beyond probability, furnishing a basis for arguing that an agent's numerical previsions, understood as fair prices, must satisfy the mathematical laws of expectation.

In view of de Finetti's construal of prevision as fair price, a significant consequence of the preceding results is that an agent's prevision function may be confined to any collection of bounded random quantities of interest to the agent at a given moment. The agent need not evaluate previsions for random quantities which do not belong to the collection, and aside from the boundedness assumption, the collection is not subject to any algebraic or topological closure constraints, measurability conditions, or subsidiary cardinality restrictions.

Nevertheless, provided the prevision function defined on the collection of interest is coherent, the agent may in principle freely extend his prevision function to a coherent prevision function also defined for each random quantity belonging to some additional collection of bounded random quantities of interest. By contrast, in accounts of probability that impose countable additivity (or more generally, monotone sequential continuity), a similar freedom to extend cannot be granted, even if the collection in question meets such restrictions.[2]

De Finetti's bookmaking approach is an honored benchmark in the foundations of probability. The driving force of this approach—subject, of course, to several structural assumptions (about which more below)—is the criterion of coherence. As a decision-theoretic requirement, the criterion of coherence enforces the *principle of uniform simple dominance*—the injunction to reject as unacceptable for choice any option uniformly simply dominated by another option available for choice. Having taken the satisfaction of the structural conditions as given, de Finetti contended that this decision-theoretic requirement is a normative standard to which previsions, understood as fair prices, ought to conform.

The constraints I have described as "structural" assumptions of the bookmaking approach are those requirements explicitly or implicitly presupposed of the system of beliefs, values, and decisions to which a given agent is committed in the decision-theoretic scheme underlying the account of coherent previsions. Although several of them are commonly taken for granted in theories of subjective probability and expected utility, these constraints are not in general rationally obligatory on all agents in all contexts, or at least not taken to be so uncontroversially. Such structural constraints include, for example, the presupposition that an agent's probability judgment about any given event does not vary with which gamble from among those gambles available he chooses to take (*act-state probabilistic independence*); the presupposition that an agent's valuation of any given outcome does not vary with which gamble and state result in the outcome (*state-outcome value independence*); and the presupposition that the agent's valuation of all possible outcomes in each state is represented by a real-valued function linear in numerical values (*real-valued linear utility*). Perhaps some of these assumptions can be construed as working hypotheses, cast as illustrative simplifications, or parlayed into substantive principles, say, in the presence of other assumptions. It is not my task here to enter into a protracted discussion about all the structural assumptions or about all the important limitations of the bookmaking approach, some of which bear general foundational or methodological significance. In this article I wish to focus on a particular line of criticism that draws attention to the normative status of the presumption of real-valued representability of uncertainty that pervades foundational accounts of subjective probability and expected utility.

In an influential paper, Shimony [1955] criticized de Finetti's account of coherent previsions, and by extension standard normative theories of subjective probability and expected utility, for allowing for violations of a more demanding dominance condition called the *principle of weak dominance*. This decision-theoretic principle enjoins an agent to reject as unacceptable for choice any option that, by his own lights, is possibly worse and certainly no better than another option open for him to choose. To insure the principle of weak dominance assumes its putative role as a normative standard, Shimony—and subsequently Kemeny [1955], Stalnaker [1970], Carnap [1971], and Skyrms [1980, 1995], among

[2] See, for example, [Banach and Kuratowski, 1929], [Ulam, 1930], and [Vitali, 1905].

others—advocated a modification of the bookmaking approach that replaces de Finetti's criterion of coherence with a stricter coherence criterion, aptly called *strict coherence*, which demands that no exchange of finitely many appraised gambles at their fair prices leads to a possible loss and to no net gain.

The resulting account of strictly coherent previsions requires each agent to attribute a positive fair price to any event of interest that he judges to be possible. Thus, under the construal of prevision—and so subjective probability—as fair price, each agent is obliged to attribute a positive probability to any event of interest that he judges to be possible, a condition called *regularity*. While de Finetti's account of coherent previsions requires each agent to attribute zero probability to any event of interest he judges to be impossible—a constraint that translates to a familiar consequence of the standard axioms of probability—the account of strictly coherent previsions imposes the additional restriction that forbids any agent from attributing zero probability to any event of interest he judges to be possible. This additional restriction, however, has become a well-known source of contention, for to insist that every agent be committed to a network of probability judgments representable by a real-valued probability function satisfying the condition of regularity is to impose ostensibly severe restrictions on what is to qualify as rationally permissible systems of probability judgment and matters of inquiry for a given agent to entertain at any given time (see §2). By mandating conformity to the principle of weak dominance, the account of strictly coherent previsions stands accused of being heavy-handed in the name of rationality.

To make room for the supposed wide array of rationally permissible systems of probability judgments compatible with the condition of regularity or, more generally, systems of prevision judgments (expectations) compatible with the principle of weak dominance (or strict coherence), various authors, including de Finetti, have suggested abandoning the presupposition that such systems be representable in terms of real-valued probabilities and expectations, instead allowing for representations in terms of mathematical probabilities—or, more generally, mathematical expectations—taking values in some one or another non-Archimedean ordered algebraic system, commonly some proper totally ordered field extension of the system of real numbers. A proper totally ordered field extension of the system of real numbers enjoys the usual properties of real-number arithmetic (e.g., commutativity and associativity of addition and multiplication, distributivity laws, additive and multiplicative inverse laws, translation and scale invariance laws, etc.). However, such a field extension is necessarily non-Archimedean, accordingly including nonzero numbers that are infinitely small, called *infinitesimals*, and by closure under multiplicative inverses, numbers that are infinitely large, called *unlimited numbers*.

A number of philosophers have advocated the proposal to allow for non-Archimedean representations of probability, or more generally, of expectation. Advocates include, for example, Lewis [1986, 1996] and Herzberg [2007], in addition to Carnap [1980] and Skyrms [1980, 1995]. Lewis and Skyrms, avid defenders of imposing regularity, write approvingly of the mathematical developments due to Bernstein and Wattenberg [1969], suggesting that a satisfactory formal theory of probability would be an outgrowth of their mathematical advances. Similarly, in several individually and jointly authored articles, Benci et al. have advanced a formal theory of probability also enforcing regularity while conceding the

Archimedean property [Benci et al., 2013, Wenmackers and Horsten, 2010, Wenmackers, 2012].

Enjoying broad expressive flexibility, non-Archimedean representations have been put to use not only in philosophy, but also in areas such as decision theory, game theory, mathematical statistics, and logic. For example, in decision theory, Hammond [1999] has explored representations of preference in terms of non-Archimedean expected utility, while in game theory, Blume et al. [1991] have appealed to such a representation they themselves established to characterize equilibrium refinements. In mathematical statistics, Cozman and Seidenfeld [2007] have studied independence concepts for non-Archimedean representations of probability. In philosophical logic, Lehmann and Magidor [1992] have investigated a non-Archimedean probability representation for conditionals. Other applications have arisen in mathematical economics and physics, to include but two more. This brief list, of course, is merely illustrative.

Despite the enthusiasm for non-Archimedean representations and the headway made by formal developments of non-Archimedean probability and expected utility, surprisingly little has been done to articulate, let alone secure, adequate foundations for a general theory of non-Archimedean probability and expectation. Of course, efforts toward this end require more than merely proposing an abstract formalism admitting models in which mathematical probabilities complying with a formal rendition of the condition of regularity—or, more generally, mathematical expectations complying with a formal rendition of the principle of weak dominance (or strict coherence)—take values in some non-Archimedean ordered algebraic system. Such efforts entail offering an account of the intended interpretation of the symbols and operations of the proposed formalism and defending the mathematical constraints imposed upon the symbols and operations of the formalism under the given interpretation.

To this end, some embracing a decision-theoretic viewpoint may suggest adapting the bookmaking approach to the case at hand by allowing fair prices to take values in some ordered algebraic system such as a proper totally ordered field extension of the system of real numbers. Questions about the intelligibility of such a conception of fair prices notwithstanding, this reformulation of the bookmaking approach makes the gratuitous assumption that the network of probability judgments (or more generally, expectations) to which a given agent is committed at a given time is representable in terms of some fixed ordered, possibly non-Archimedean, algebraic system, a constraint whose formulation and imposition are presumably at issue in trying to secure acceptable normative foundations. In general, proposed formalisms that furnish even so much as an account of mathematical expectation not only impose substantive cardinality restrictions on the mathematical structures subsumed under them (e.g., by requiring state spaces to be finite or countable) but also impose weakened Archimedean constraints by way of, for example, postulating special topological or continuity conditions relative to some one or another fixed ordered algebraic system, encoding what is tantamount to the gratuitous assumption of the reformulated bookmaking approach. In short, existing accounts of non-Archimedean probability and expectation proposed by philosophers and non-philosophers alike deliver mathematically deficient formalisms and rest on inadequate theoretical foundations.

In this article, I shall take a step toward rectifying this predicament. Following some formal preliminaries (§1) and a more detailed critical review of de Finetti's bookmaking approach to the foundations of probability (§2), I shall recast de Finetti's quantitative account of coherent previsions within an analytic framework in which an agent's *qualitative* comparative prevision judgments are regulated by a *qualitative reformulation* of his criterion of coherence (§3). I show that *comparative previsions* satisfy *qualitative analogues* of de Finetti's famous bookmaking theorem and his Fundamental Theorem of Prevision.

I shall thereupon introduce an account of expectation based on a different qualitative criterion of coherence for qualitative comparisons between gambles (§4). As with the qualitative reformulation of de Finetti's account, the qualitative comparisons are interpreted as an agent's (all things considered) comparative preference judgments over gambles or more directly as an agent's comparative expectation judgments over random quantities. The alternative coherence criterion regulating an agent's rational *comparative expectations* is reminiscent of de Finetti's criterion of coherence, yet, as I explain, it does not impose an Archimedean condition on qualitative comparisons. Moreover, unlike de Finetti's criterion of coherence, the qualitative criterion I propose respects a qualitative version of the principle of weak dominance under its intended interpretation. Just as de Finetti formulated his criterion of coherence for an *arbitrary* collection of *bounded* gambles, free of measurability, closure, and subsidiary cardinality conditions, I formulate the qualitative criterion of coherence for an arbitrary collection of gambles. In fact, I do not even require the quantities to be *bounded*, the dispensation of which permits the collection to be truly *any* collection of gambles of interest to the agent. Furthermore, in contrast with many developments of comparative probability, the qualitative criterion of coherence I advance does not require the binary relation reflecting an agent's comparative judgments to satisfy *reflexivity, completeness*, or even *transitivity*.

In spite of these weak structural assumptions, I shall describe a result asserting that any coherent system of comparative expectations can be extended to a coherent system of comparative expectations also specifying a comparison between two additional gambles of interest. Thus, this result, Theorem 5.2 of §5, is a qualitative analogue of de Finetti's Fundamental Theorem of Prevision. As with de Finetti's result, a consequence of this *Fundamental Theorem of Comparative Expectations*, reported in Corollary 5.4 of §5, is that any coherent system of comparative expectations can be extended to a weakly ordered coherent system of comparative expectations over any collection of gambles containing the initial collection of gambles of interest. In addition to the qualitative version of the principle of weak dominance, the coherent weakly ordered system of comparative expectations satisfies familiar additivity and scale invariance postulates (i.e., the independence axiom) when the extended collection forms a linear space. Of course, the results of this article apply to the special case of comparative judgments between (indicator functions of) events and thus to the case of comparative probability judgments.

I remark at the outset that although in this article I shall assume that the outcomes of gambles are specified in terms of real numbers, the results herein remain valid, of course suitably reformulated, if instead an agent's valuation over outcomes is representable in terms of elements of, say, a partially ordered Abelian group (or preordered Abelian group), thus admitting non-Archimedean preferences over outcomes. In particular, an agent's valuation over outcomes may be representable in terms of numbers in some (possibly non-Archimedean) totally ordered field extension of the system of real (or rational) numbers.

The technical developments of this article complement those in another article [Pedersen, 2013] where in addition to formulating a generalized concept of a system of comparative expectations, I establish a numerical representation theorem for a system of comparative expectations in the generalized sense. It follows from the results proven for the general framework of the companion article that any coherent weakly ordered system of comparative expectations on a unital linear space over a totally ordered field can be represented by a normalized, linear function taking values in a (possibly non-Archimedean) totally ordered field extension of the system of real (or rational) numbers—that is, an *expectation function*—that respects the principle of weak dominance according to the intended interpretation of the analytic framework. The general account permits non-Archimedean preferences over outcomes. As a result, the technical developments of these articles contain key ingredients of an unconditionally *full* account of subjective expected utility in the style of Savage [1954] and Anscombe and Aumann [1963].

Instead of adopting a routine ultraproduct construction to cobble together the totally ordered field extension, the proof I offer for the numerical representation theorem extends and thereupon employs a version of the *Hahn Embedding Theorem* to ultimately construct the desired totally ordered field in which the expectation function is to take values. As among other things I explain in more detail in [Pedersen, 2013], this route affords an expression of numerical values in terms of *formal power series* in a *single* infinitesimal and enables infinitely large numerical differences in expectations of gambles to be explicitly *traced* to infinitely large qualitative differences in comparisons of these gambles, since the ordered collection of the powers of the single infinitesimal belonging to the image of the expectation function is order-isomorphic to the ordered collection of *Archimedean equivalence classes* of gambles from the system of comparative expectations. Thus, the route by way of the Hahn Embedding Theorem usefully locates the origins of numbers of significant interest. I shall state a corollary and further discuss the numerical representation theorem in §6.

An account of subjective probability and expected utility that rests upon qualitative criteria regulating preferences (or comparative judgments of expectation) enjoys an important polemical vantage from a foundational viewpoint. Like other accounts such as Savage's theory of personal probability, the account I advance in this article and in [Pedersen, 2013] does not presume that an agent can or should make numerical judgments of probability or expectation. Nonetheless, any coherent system of comparative expectations can be extended to a coherent weakly ordered system of comparative expectations that is numerically representable by an expectation function. The uniqueness and invariance properties of such a numerical representation facilitate a systematic investigation of my account. In addition, since the qualitative criterion of coherence I advance does not presuppose that a given agent's comparative judgments are reflexive, complete, or even transitive, my account furnishes a basis for theory admitting representations in terms of numerically imprecise, and possibly non-Archimedean, probabilities and expected utilities in the style of, for example, Walley [1991] and Williams [1975] (see also Levi [1974] and Smith [1961]).

1. Formal Preliminaries

Throughout set-theoretic notation is standard. Herein I introduce additional notation and terminology I shall use throughout the article and review some formal concepts.

A mapping $f : S \to T$ is said to be a *constant function* if there is $t \in T$ such that $f(s) = t$ for every $s \in S$; following usual practice, a bold $\boldsymbol{t}$ shall denote a constant function assigning t to each s in S. Following ordinary convention, if a set T has been fixed in discussion, $S \subseteq T$, and the context is clear, S shall be identified with its (T-relative) *indicator function*, a mapping on T, again denoted by S, such that $S(t) = 1$ if $t \in S$ and $S(t) = 0$ if $t \in T \backslash S$. Thus, an event shall be identified with its indicator function.

Recall that a binary relation R on a set X is called a *weak order* if it is complete and transitive; called a *preorder* if it is reflexive and transitive; called a *total order*, or simply an *order*, if it is an antisymmetric weak order; and called a *partial order* if it is an antisymmetric preorder. As usual, the pair (X, R) is called *transitive* (*reflexive*, a *weak order*, etc.) if R is transitive (reflexive, a weak order, etc.). In general, for a binary relation R on a set X, $A \subseteq X$, and $x \in X$, let $A_{Rx} := \{a \in A : aRx\}$. For example, $\mathbb{N}_{>0}$ denotes the set of positive integers. When there is no danger of confusion, I shall write A^n_{Rx} rather than $(A_{Rx})^n$, thus dropping the disambiguating parentheses.

Given a binary relation $\gtrapprox$ on a set X and $s, s' \in X$, I shall write $s \succ s'$ if $s \gtrapprox s'$ and $s' \not\gtrapprox s$, and I shall write $s \approx s'$ if both $s \gtrapprox s'$ and $s' \gtrapprox s$. Now let $\gtrapprox$ be a binary relation on X, and let $\gtrapprox'$ be a binary relation on Y. Say that the pair $(Y, \gtrapprox')$ is an *extension* of $(X, \gtrapprox)$ and that $(Y, \gtrapprox')$ *extends* $(X, \gtrapprox)$ if $Y \supseteq X$, $\gtrapprox' \supseteq \gtrapprox$, and $\succ' \supseteq \succ$. The notation $\gtrapprox$ shall be used for preferences (or qualitative comparisons); I shall also use $\gtrsim$ for this purpose with similar notational abbreviations $\succ$ and $\sim$.

Recall that a *totally ordered field* is a sextuple $(\mathbb{F}, +, \cdot, 0, 1, \geqslant)$ equipped with a commutative and associative addition operation $+$ on $\mathbb{F}$ with an additive identity 0 such that every element of $\mathbb{F}$ has an additive inverse, a commutative and associative multiplication operation $\cdot$ on $\mathbb{F}$ with a multiplicative identity 1 such that multiplication distributes over addition and every nonzero element of $\mathbb{F}$ has a multiplicative inverse, and a total order $\geqslant$ on $\mathbb{F}$ satisfying the property that $r \geqslant s$ implies $r + t \geqslant s + t$ and the property that $r \geqslant 0$ and $s \geqslant 0$ imply $r \cdot s \geqslant 0$ (for all $r, s, t \in \mathbb{F}$). Following usual practice, and of course when the context is clear, the multiplication operator $\cdot$ shall be suppressed, and the set $\mathbb{F}$ shall be itself called a (totally ordered) field. Recall that a totally ordered *subfield* of a totally ordered field $\mathbb{F}$ is a subset of $\mathbb{F}$ equipped with the field operations, additive and multiplicative identities, and a total order inherited from $\mathbb{F}$. A totally ordered field $\mathbb{F}$ is a said to be an *extension* of a totally ordered field $\mathbb{K}$ if $\mathbb{K}$ is a totally ordered subfield of $\mathbb{F}$.

Let S be a set. The set of mappings $\mathbb{R}^S$ forms a (real) vector (linear) space under the naturally inherited *pointwise* operations of addition $+$ and scalar multiplication $\cdot$ defined by setting $(r \cdot f)(s) := r \cdot f(s)$ and $(f + g)(s) := f(s) + g(s)$ for all $r \in \mathbb{R}$, $f, g \in \mathbb{R}^S$, and $s \in S$ (with the notation for real addition and multiplication unchanged). As usual, the notation for multiplication $\cdot$ shall be suppressed when there is no danger of confusion. Let $\mathbb{F}$ be a totally ordered subfield of $\mathbb{R}$ (e.g., $\mathbb{Q}$). An $\mathbb{F}$-*linear space* of $\mathbb{R}^S$ shall refer to a subset $\mathscr{L}$ of $\mathbb{R}^S$ such that $rf + tg \in \mathscr{L}$ whenever $f, g \in \mathscr{L}$ and $r, t \in \mathbb{F}$. An $\mathbb{F}$-linear space $\mathscr{L}$ of $\mathbb{R}^S$ shall be called *unital* if $\mathbf{1} \in \mathscr{L}$.

Again, let S be a set, and let $f, g \in \mathbb{R}^S$. The notation $f \geqslant g$ shall express that $f(s) \geqslant g(s)$ for each $s \in S$, whereby the function f is said to *dominate* the function g. The notation $f \gg g$ shall express that $f(s) > g(s)$ for each $s \in S$, whereby the function f is said to *simply dominate* the function g. Say that the function f *uniformly simply dominates* the function g, and write $f \ggg g$, if there is $\epsilon \in \mathbb{R}_{>0}$ such that $f \gg g + \boldsymbol{\epsilon}$. Finally, say that the function f *weakly dominates* the function g, and write $f > g$, if $f(s) \geqslant g(s)$ for each $s \in S$ and $f(s) > g(s)$ for some $s \in S$. So much for the formal preliminaries.

2. De Finetti's Criterion of Coherence

De Finetti's bookmaking approach to the foundations of probability seeks firm theoretical grounds on the basis of the criterion of coherence. The purpose of this section is to review de Finetti's somewhat informal account and the criticisms discussed in the introduction. In the following section, I shall recast de Finetti's account in the formal framework of the present article.

Let Ω be a fixed set of *states* for a given agent at a given time. The set of states corresponds to a family of propositions which are individually consistent, mutually exclusive, and collectively exhaustive relative to the agent's state of information at the given time. A subset (or its indicator function) of Ω is called an *event*. In de Finetti's account, an *outcome*, represented by a real number, specifies an amount of a single currency denominated in units of some personally valued commodity over which utility is linear, such as money or even utiles themselves. A function $f : \Omega \to \mathbb{R}$ is called a *gamble* (or *random quantity*). Accordingly, given a gamble f and state ω, the value $f(\omega)$ represents the outcome to result if gamble f has been chosen for implementation and state ω has become known to be the case.[3] It is assumed that an agent's decision to implement a gamble does not alter his probability judgment about any given event (*act-state probabilistic independence*) and moreover that the agent's valuation of any given outcome is independent of which gamble and state result in the outcome (*state-outcome value independence*).

Let $\mathscr{X}$ be a fixed collection of bounded gambles of interest to the agent at the given time.[4] De Finetti presumed that for each gamble f in $\mathscr{X}$ there is a unique real number $\mathbb{P}(f)$ specifying the *fair price* at which the agent is indifferent between receiving (giving) the gamble f and receiving (giving) the amount $\mathbb{P}(f)$. More fully, acting as *bookie*, the agent is presumed to have adopted a *price function* $\mathbb{P} : \mathscr{X} \to \mathbb{R}$ whereby he is committed to honor the demand of any *opponent* calling upon him to exchange finitely many gambles $(f_i)_{i=1}^n$

[3]De Finetti took gambles as primitive elements of his theory and in particular did not fix a set of states Ω. Nonetheless, as de Finetti himself described, it is possible to construct a space of personal possibilities for a given set of random quantities of interest. For simplicity of exposition, I shall assume that this has been done. See [de Finetti, 1974a, Chapter 2] and [de Finetti, 1974b, Appendix] for detailed discussion.

[4]Naturally, a gamble f is *bounded* if there is a number $n \in \mathbb{N}$ such that $|f(\omega)| \leqslant n$ for every $\omega \in \Omega$. Albeit less systematically, de Finetti also considered the case in which random quantities are unbounded [de Finetti, 1974a, pp. 130-132, pp. 245-252]. For recent work related to systemic developments in this direction for (possibly extended) real-valued expectations, see, for example, [Berti et al., 2001], [Crisma et al., 1997], and [Schervish et al., 2009, pp. 9-18]. See also [Seidenfeld et al., 2009, pp. 334-335] for a discussion of this case in the context of an exploration of unbounded utilities. In the account to be developed in this article, I impose no boundedness requirement on gambles.

from $\mathscr{X}$ in respective real-valued units $(c_i)_{i=1}^n$ at their respective prices $\big(\mathbb{P}(f_i)\big)_{i=1}^n$, thereby exchanging, for each $i = 1, \ldots, n$, the gamble $c_i f_i$ for the gamble $c_i\mathbb{P}(f_i)$, rather than taking his alternative option to abstain from such an exchange of gambles, which results in a sure outcome of 0. The values c_i may be positive, negative, or zero, determining the units and direction of the exchange of f_i at price $\mathbb{P}(f_i)$. So if $c_i > 0$, the agent must *give* the opponent $c_i\mathbb{P}(f_i)$ in order to *receive* $c_i f$, and if $c_i < 0$, the agent must *give* the opponent $-c_i f$ in order to *receive* $-c_i\mathbb{P}(f_i)$. The resulting net outcome of an individual exchange, $c_i(f_i - \mathbb{P}(f_i))$, may be viewed as the random outcome of a *fair bet* to which the agent, acting as bookie, is committed to accept in place of abstaining from betting altogether.

Within this betting scheme, de Finetti articulated his criterion of coherence for gambling. This criterion requires that no finite series of fair bets be uniformly simply dominated by abstaining. De Finetti aimed to justify the mathematical laws of expectation and thus probability by dint of the authority accorded to the criterion of coherence in rational decision making (see §1 for notational conventions and terminology).[5]

DEFINITION 2.1. Let $\mathscr{X}$ be a collection of bounded gambles on a set of states Ω. A price function $\mathbb{P}$ on $\mathscr{X}$ is said to be *coherent* if there is no $n \in \mathbb{N}_{>0}$ such that for some gambles $(f_i)_{i=1}^n \in \mathscr{X}^n$ and real-valued units $(c_i)_{i=1}^n \in \mathbb{R}^n$:

$$\sum_{i=1}^{n} c_i\Big(f_i - \mathbb{P}(f_i)\Big) \quad \lll \quad \mathbf{0}.$$

Otherwise, $\mathbb{P}$ is said to be *incoherent*. ◂

Thus, a price function $\mathbb{P}$ on $\mathscr{X}$ is incoherent if there is a finite sequence of appraised gambles $f_1, \ldots, f_n$ and real-valued units $c_1, \ldots, c_n$ such that the agent is prepared to exchange, for each $i = 1, \ldots, n$, the gamble $c_i f_i$ for the gamble $c_i\mathbb{P}(f_i)$, resulting in a position in which abstaining from such a series of individual exchanges *uniformly simply dominates* the random net outcome $\sum_{i=1}^n c_i(f_i - \mathbb{P}(f_i))$ obtained from making the series of exchanges. In such a case, the agent has laid down bets which he evaluates as fair but which all together *certainly* result in a uniform loss to him, and is accordingly said to be in *Dutch Book*, or simply *book*.

[5] De Finetti also articulated a criterion of conditional coherence for conditional previsions. Rather than defining conditional prevision in terms of the usual ratio formula, de Finetti introduced conditional prevision as a primitive concept applying to what he called *conditional gambles* (or *conditional random quantities*) (see, e.g., [de Finetti, 1974a, p. 62, pp. 139-140]). The criterion of conditional coherence allows for the expectation of a conditional gamble to take an *arbitrary value* conditional on a zero probability event, leading de Finetti to propose an axiom to adjoin the other axioms of conditional probability in order to admit and to constrain evaluations conditional on zero probability events; he also informally suggested an associated strengthened criterion of coherence (see, e.g., [de Finetti, 1949, pp. 81-83], [de Finetti, 1974b, pp. 338-340, 344-348]). Subsequently, various authors, such as Holzer [1985] and Regazzini [1985], have endorsed and explored de Finetti's modified conception of a conditional expectation. See [Cifarelli and Regazzini, 1996] for references to recent work as well as [Gilio and Sanfilippo, 2013] of the present issue for a discussion of conditional random quantities.

De Finetti showed that any coherent price function satisfies the properties of an ordinary *finitely additive expectation*. That is, any coherent prevision function $\mathbb{P}$ on $\mathscr{X}$ satisfies the following properties for all gambles f, g on Ω and real numbers r, s in $\mathbb{R}$:

(P1) $\mathbb{P}(\mathbf{1}) = 1$ if $\mathbf{1} \in \mathscr{X}$;

(P2) $\mathbb{P}(rf + sg) = r\mathbb{P}(f) + s\mathbb{P}(g)$ if $f, g, rf + sg \in \mathscr{X}$;

(P3) If $f \ggg g$, then $\mathbb{P}(f) > \mathbb{P}(g)$, if $f, g \in \mathscr{X}$.

Property P1 is called *normalization*, and property P2 is called *linearity*. Property P3 is a numerical expression of a dominance principle discussed in the introduction called the *principle of uniform simple dominance*. Property P3 demands that if with respect to each state in Ω, gamble g is worse than gamble f even when supplemented with a fixed positive amount ϵ, then the fair price of g ought to be less than the fair price of f. Indeed, if the numerical ordering of the agent's fair prices represents his preference ranking over gambles, then this property aligns with a reasonable requirement of rational decision making: If f uniformly simply dominates g—that is, $f \gg g + \boldsymbol{\epsilon}$ for some real number $\epsilon > 0$—then g ought to be rejected for choice when f is available for the agent to choose.

Summarizing, de Finetti proved the following result [1974a, pp. 69ff.] (cf. [1949, pp. 105-106], [1974b, p. 335]).[6]

FACT 2.2 (de Finetti 1949, 1974a,b). *Let $\mathscr{X}$ be a collection of bounded gambles on a set of states Ω, and let $\mathbb{P}$ be a price function on $\mathscr{X}$. Then if $\mathbb{P}$ is coherent, it is a finitely additive expectation.* ■

As I indicated in the introduction, de Finetti also proved an important extension theorem called the *Fundamental Theorem of Prevision* [1974b, pp. 336-338] (cf. [1949, pp. 105-107], [1974a, pp. 111-116]). To state this theorem, I introduce some notation. Given a collection of bounded gambles $\mathscr{X}$, a bounded gamble f, and a price function $\mathbb{P} : \mathscr{X} \to \mathbb{R}$, let $\mathfrak{L}(\mathbb{P}, f)$ denote the following set:

$$\left\{r \in \mathbb{R} : r = c_0 + \sum_{i=1}^{n} c_i \mathbb{P}(g_i) \text{ for some } n \in \mathbb{N}_{>0}, (c_i)_{i=0}^n \in \mathbb{R}^{n+1}, (g_i)_{i=1}^n \in \mathscr{X}^n \text{ with } c_0 + \sum_{i=1}^{n} c_i g_i \leqslant f\right\}.$$

Similarly, let $\mathfrak{U}(\mathbb{P}, f)$ denote the following set:

$$\left\{r \in \mathbb{R} : r = c_0 + \sum_{i=1}^{n} c_i \mathbb{P}(g_i) \text{ for some } n \in \mathbb{N}_{>0}, (c_i)_{i=0}^n \in \mathbb{R}^{n+1}, (g_i)_{i=1}^n \in \mathscr{X}^n \text{ with } f \leqslant c_0 + \sum_{i=1}^{n} c_i g_i\right\}.$$

[6] De Finetti focused on property (P2) and the requirement that $\inf\{f(\omega) : \omega \in \Omega\} \leqslant \mathbb{P}(f) \leqslant \sup\{f(\omega) : \omega \in \Omega\}$ for every $f \in \mathscr{X}$, which implies property (P1). Clearly de Finetti's criterion of coherence also implies property (P3). I have formulated de Finetti's result in terms of a finitely additive expectation not only to align it with the standard concept of a normalized, order-preserving linear functional but also to draw attention to the separate roles of normalization, linearity, and the principle of uniform simple dominance.

FACT 2.3 (de Finetti 1949, 1974a,b). *Let $\mathscr{X}$ be a collection of bounded gambles on a set of states Ω, and let $\mathbb{P}$ be a coherent price function on $\mathscr{X}$. Then*:

(i) *If f is a bounded gamble on Ω, then*:
  (a) *$\mathbb{P}$ can be extended to a coherent price function on $\mathscr{X} \cup \{f\}$, and*
  (b) *If $\widehat{\mathbb{P}}$ is an extension of $\mathbb{P}$ to $\mathscr{X} \cup \{f\}$, then $\widehat{\mathbb{P}}$ is coherent if and only if $\widehat{\mathbb{P}}(f) \in [\sup \mathfrak{L}(\mathbb{P}, f), \inf \mathfrak{U}(\mathbb{P}, f)]$*;

(ii) *If $\mathscr{Y}$ is a collection of bounded gambles on Ω containing $\mathscr{X}$, then $\mathbb{P}$ can be extended to a coherent price function on $\mathscr{Y}$.* ∎

Part (i) is de Finetti's Fundamental Theorem of Prevision. Part (ii) is a corollary of this theorem obtained by a routine application of Zorn's Lemma.

Where $\mathscr{X}$ is an algebra of events, the two results together entail the celebrated *Dutch Book Theorem*. Thus, based on the criterion of coherence, it is argued that an agent's numerical credences, understood as fair prices, must conform to the mathematical laws of probability. Yet as I mentioned in the introduction, Fact 2.2 and Fact 2.3 go well beyond probability, forming the basis of an argument that an agent's numerical previsions, understood as fair prices, must conform to the mathematical laws of expectation. As we have seen, this justification rests on an assumed relationship with rational decision making. Coherence derives its normative status in virtue of its respect for the principle of uniform simple dominance, an injunction that rules out options, identifying those options which ought to be rejected for choice.

As I indicated in the introduction, various authors have endorsed a strengthened dominance principle called the *principle of weak dominance*. In the present context, the following condition is a numerical expression of the principle of weak dominance (see §1 for notational conventions and terminology):

(P4) If $f > g$, then $\mathbb{P}(f) > \mathbb{P}(g)$, if $f, g \in \mathscr{X}$.

Much like property P3, property P4 aligns with a compelling norm of rational decision making: If f weakly dominates g—that is, the agent judges that g is certainly no better and possibly even worse than f—then the agent ought to reject the gamble g as unacceptable for choice when the gamble f is also available as an option. On the basis of a putative obligation to respect the principle of weak dominance, numerous authors have criticized de Finetti's account, and so by extension standard theories of subjective probability and expected utility, for not observing it. Shimony, for example, writes:

> Ramsey's and DeFinetti's notion of coherence seems too weak, however. There are sets of beliefs which are classified as coherent by their definition, but which intuitively we should classify as incoherent. Specifically, suppose X's beliefs are such that an opponent can propose a series of bets acceptable to X on the basis of his beliefs, and such that (i) X does not suffer a net loss in every eventuality, yet (ii) he makes a net gain in no eventuality, and in at least one possible eventuality he suffers a net loss. X's beliefs in this example are coherent, according to Ramsey's and DeFinetti's notion of coherence, although intuitively we are inclined to say that they are incoherent [1955, pp. 8-9].

For previsions restricted to events, various defenders of the principle of weak dominance have advocated a stronger coherence criterion called *strict coherence*, which forbids any finite series of fair bets from being weakly dominated by abstaining. The following definition applies more generally to bounded gambles.

DEFINITION 2.4. Let $\mathscr{X}$ be a collection of bounded gambles on a set of states Ω. A price function $\mathbb{P}$ on $\mathscr{X}$ is said to be *strictly coherent* if there is no $n \in \mathbb{N}_{>0}$ such that for some gambles $(f_i)_{i=1}^n \in \mathscr{X}^n$ and real-valued units $(c_i)_{i=1}^n \in \mathbb{R}^n$:

$$\sum_{i=1}^{n} c_i\Big(f_i - \mathbb{P}(f_i)\Big) \quad < \quad \mathbf{0}.$$

Otherwise, $\mathbb{P}$ is said to be *strictly incoherent*. ◂

Shimony thought that strict coherence can stand by itself, claiming that it is analytic, or "true in virtue of relations which hold between the concepts of confirmation and the concepts of coherence, rather than in virtue of any contingent facts" [1955, p. 10]. So too have other authors held the criterion of strict coherence in high esteem, exponents of which include Carnap [1971, p. 114], Kemeny [1955, p. 264, pp. 272-273], Stalnaker [1970, pp. 67-68], and Skyrms [1980, pp. 73-74] (cf. [1995, pp. 39-40]). Yet in addition to the usual laws of finitely additive probabilities, with the criterion of strict coherence comes a strong requirement on an agent's numerical credences relative to his state of information, for as mentioned in the introduction, strict coherence forbids the agent from attributing zero probability to any event of interest that he judges to be possible. This is easy to see: If $\mathbb{P}(E) = 0$ for some event E judged to be possible, then $\mathbb{P}(E) - E$ has a negative outcome in some states and a positive outcome in no states, contrary to the criterion of strict coherence. Thus, subject to the criterion of strict coherence, each event of interest that is judged to be possible must be attributed positive probability, a property called *regularity*.

While some authors have found regularity compelling, others have found it too restrictive. After all, no real-valued probability function can assign positive probability to more than denumerably many pairwise disjoint events, notwithstanding that a uniform real-valued probability over denumerably many pairwise disjoint events must assign zero probability to each such event. Thus regularity rules out multitudes of formal models from mathematical probability, statistics, and other sciences if rational credences are required to be representable by real-valued probabilities (representing rational credences in terms of continuous distributions is forbidden!). While regularity prohibits agents from inquiring about more than denumerably many pairwise disjoint events for attributing real-valued probabilities, regularity also prohibits agents from inquiring about only finitely many events for attributing real-valued probabilities when in given states of information attributing zero real-valued probability to some of the events of interest is seemingly rationally unobjectionable subject to the requirement of real-valued representability (e.g., finite state spaces which naturally admit refinements to infinite state spaces, as when, say, an agent's state of information cannot rule out arbitrarily deep nested subdecompositions of a finite decomposition of a dartboard; similarly, an agent who knows finitely many values of some function may occupy a state of information that cannot rule out infinitely many different forms the function might take; or, again, an agent who knows that a coin will be flipped until it lands heads for the first time occupies a state of information that cannot rule out that the coin will never land heads). Put differently, if judgments of probability are required to be representable by real-valued probabilities satisfying the condition of regularity, then any agent inquiring about finitely many events must attribute positive probability to each event of interest even if attributing zero probability to some of the events of interest would not be unreasonable in his given state of information. As such, requiring that systems of probability judgments be representable by real-valued probability functions conforming to the requirement of regularity is to place severe restrictions on what systems of probability judgments and matters of inquiry are rationally permissible for agents to entertain.

Among those who find regularity too restrictive is de Finetti himself, since he advocated a theory admitting all finitely additive previsions, in particular one admitting previsions assigning a real value of zero to events judged possible. Nonetheless, even de Finetti expressed ambivalence about the apparent tension between strict coherence and regularity in his less guarded moments, conceding to the rational obligation to comply with the principle of weak dominance while at the same time conceding to the constraints of the mathematical representation of prevision in terms of real numbers, and even suggesting non-Archimedean representations for probability (see, e.g., [1931, p. 316], [1949, p. 82], [1974b, pp. 344-348, pp. 361-367], [2008, p. 118-122]).

Additionally, as de Finetti was well-aware, his theory of prevision violates an even weaker property, a numerical expression of a dominance principle called the *principle of simple dominance* (see §1 for notational conventions and terminology):

(P5) If $f \gg g$, then $\mathbb{P}(f) > \mathbb{P}(g)$, if $f, g \in \mathscr{X}$.

This condition, which can be no more compelling than the numerical principle of weak dominance, is also stronger than the numerical principle of uniform simple dominance. Yet if de Finetti's theory did not violate the principle of simple dominance, it would preclude some finitely additive previsions, a circumstance that he did not desire, as illustrated by the following simple example due to de Finetti himself (cf. [1949, p. 77], [1974a, p. 120]).

EXAMPLE 2.5 (Integer Lottery). An agent is certain that the true value of a quantity f of interest has the form $-\frac{1}{n}$ for each positive $n \in \mathbb{N}$. Given his state of information, the agent's numerical credence that the quantity will take any given one of these values is the same. Thus, the agent must assign 0 real-valued probability to each of the possible values of f, and he is obliged to expect the value of f to be 0. ♦

In more detail, let $\mathbb{P}$ be a finitely additive probability function on the positive natural numbers such that $\mathbb{P}(\{n\}) = 0$ for every positive $n \in \mathbb{N}$, and let f be a random quantity such that $f(n) = -\frac{1}{n}$ for every positive $n \in \mathbb{N}$. Then $\mathbb{P}$ is coherent, $\mathbb{P}(f) = 0$, yet $f \ll \mathbf{0}$.[7] Thus the bet $f - \mathbb{P}(f)$ would be *simply dominated* by $\mathbf{0}$. So if the stronger principle of simple dominance were adopted in place of the principle of uniform simple dominance in a modification of de Finetti's criterion of coherence, giving rise to a stronger criterion of coherence, then by definition $\mathbb{P}$ would be incoherent, thereby excluding some finitely additive probabilities.

It is all too easy to dismiss the foregoing problems on the grounds that they are mere artifacts of mathematical idealizations involving abstract systems such as lotteries with infinitely many tickets or dartboards composed of infinitely many points that can be occupied by darts with infinitely precise tips, systems that no one believes *really* exist anyway. While it may be true that you are certain that such systems do not exist, that nearly everyone you know is certain that such systems do not exist, and that as a matter of fact they do not exist, this is beside the point. As Levi [1989] has correctly pointed out, what must be shown is that any agent who embraces a state of information countenancing such an abstract system violates minimal standards of internally consistent judgements of certainty. Excluding states of information that countenance such systems would entail introducing substantive constraints on a weak conception of rational states of information.

That said, we *can* and often *do* embrace, contemplate, and reason about such "idealized" systems. To be sure, we accept the formal consequences of our best scientific theories in order to make probabilistic estimates about various quantities of theoretical or practical interest. We often take for granted that such quantities are continuous, even when only interval-valued approximations for measurements are available. Even for relatively insignificant endeavors, we embrace systems like

[7] Observe that by coherence $0 \geqslant \mathbb{P}(f) = \mathbb{P}(f \cdot \{n : n \geqslant m\}) \geqslant -\frac{1}{m}$ for every positive $m \in \mathbb{N}$, so $\mathbb{P}(f) = 0$.

"idealized" dartboards as felicitous representations of our information for conducting our reasoning and deliberations.

The position I investigate in this article affirms that independently of considerations of numerical probability, an agent ought to be committed to respecting the principle of weak dominance: An agent ought to reject an option that is *certainly* no better and *possibly* worse than an another option available for him to choose. Accordingly, in the betting context, an agent who has appraised fair prices for a collection of gambles has thereby made a commitment that should preclude him from judging a finite series of fair bets as acceptable for choice in a given decision in which his alternative option to abstain weakly dominates making the series of fair bets. Similar remarks apply to the principle of simple dominance and of course the principle of uniform simple dominance. At the same time, the position I investigate in this article holds that every agent ought to be allowed to embrace any network of judgments of certainty, probability, expectation, value, and choice concerning any matter of inquiry whatsoever subject to the requirement that the agent's attitudes conform to minimal standards of coherence or internal consistency. It is well-known that these desiderata pull in different directions under the presumption of real-valued representability of uncertainty, and an oft-proposed resolution to the tension is to adopt a suitably flexible framework admitting non-Archimedean representations for probability and more generally expected utility.

3. Coherent Comparative Previsions

De Finetti's informal account of coherent previsions may be reformulated within a framework in which an agent's implicit preferential comparisons between gambles are made explicit. In this section I shall recast de Finetti's account in the formal framework of this article not only to motivate the account I shall offer in the next section in an effort to address the problems discussed in the previous section but also to facilitate a comparison between our accounts.

3.1. **Three Preliminary Observations.** Before recasting de Finetti's account, I draw attention to three observations about his original formulation that serve to guide the reformulation I shall offer.

Let $\mathscr{X}$ be a collection of bounded gambles (random quantities) on a set of states Ω. First, observe that the scalar coefficients in de Finetti's criterion of coherence may be dropped, as de Finetti himself was seemingly aware despite never plainly stating as much [1974a, pp. 74-75, fn. †] (cf. [1955, pp. 118-119], [1974b, pp. 335-336]). More precisely, de Finetti's criterion of coherence is logically equivalent to the requirement that there are no positive $m, n \in \mathbb{N}_{>0}$ such that for some gambles $(f_i)_{i=1}^m \in \mathscr{X}^m$ and $(g_j)_{j=1}^n \in \mathscr{X}^n$:

$$\sum_{i=1}^{m}\Big(f_i - \mathbb{P}(f_i)\Big) - \sum_{j=1}^{n}\Big(g_j - \mathbb{P}(g_j)\Big) \lll \mathbf{0}.$$

This simplification of de Finetti's criterion of coherence is in turn clearly logically equivalent to the criterion in the following proposition.

Proposition 3.2. *Let $\mathscr{X}$ be a collection of bounded gambles on a set of states Ω. A price function $\mathbb{P}$ on $\mathscr{X}$ is coherent if and only if it it satisfies the following condition*:

(dF$\mathbb{Z}$) *There are no positive $m, n \in \mathbb{N}_{>0}$ such that for some gambles $(f_i)_{i=1}^m \in \mathscr{X}^m$ and $(g_j)_{j=1}^n \in \mathscr{X}^n$:*

$$(*) \qquad \sum_{i=1}^{m}\mathbb{P}(f_i) + \sum_{j=1}^{n} g_j \ggg \sum_{i=1}^{m} f_i + \sum_{j=1}^{n}\mathbb{P}(g_j).$$

∎

The above inequality says that the gamble $\sum_{i=1}^{m} \mathbb{P}(f_i) + \sum_{j=1}^{n} g_j$ uniformly simply dominates the gamble $\sum_{i=1}^{m} f_i + \sum_{j=1}^{n} \mathbb{P}(g_j)$.

Second, observe that de Finetti presumed, as he himself recognized, that for each gamble f from $\mathscr{X}$, the agent in question is indifferent between gamble f and its fair price $\mathbb{P}(f)$, accordingly stipulating that the agent is committed to preferences (viz., indifferences) between gambles and their fair prices. Thus de Finetti effectively assumed that the collection of gambles includes all constant gambles corresponding to the agent's appraised fair prices. Indeed, third, observe that de Finetti presumed, again as he himself recognized, that the agent is committed to complete preferences over gambles and their corresponding fair prices according to the natural numerical ordering over outcomes [1974a, pp. 73-74, fn. †] (cf. [1970, p. 132], [1974b, pp. 335-336]).

3.3. **A Qualitative Reformulation of de Finetti's Account of Coherent Prevision.** In light of the foregoing considerations, de Finetti's theoretical account may be vividly recast as follows. Let $\gtrapprox$ be a binary relation on the joint collection $\mathcal{G}$ composed of a set of bounded gambles $\mathscr{X}$ of interest to the agent together with a set of constant gambles $\mathscr{C}$ (i.e., a set of constant real-valued functions on Ω) intended to represent the agent's appraised fair prices for gambles from $\mathscr{X}$. When $\gtrapprox$ is understood to reflect the agent's preference judgments over gambles, the expression $f \gtrapprox g$ is to be read, "f is weakly preferred to g," and to be interpreted to mean that the agent either strictly prefers gamble f to gamble g or else is indifferent between them. When $\gtrapprox$ is understood as the agent's comparative judgments of expectation, the expression $f \gtrapprox g$ shall be interpreted as the agent's expectation that the unknown quantity f is greater than or equal to the unknown quantity g.

It would be sufficient to assume that $\mathscr{C} \subseteq \mathscr{X}$, accordingly presuming what can be achieved on the basis of the reformulation that is to follow presently. Here, however, I shall explicitly distinguish between the collection of gambles $\mathscr{X}$ of interest to the agent and a set of constant functions $\mathscr{C}$ intended to represent the agent's appraised fair prices in order to make the relationship to de Finetti's account more conspicuous.[8] Consider the following definition stating the conditions regulating the agent's comparative judgments (see §1 for notational conventions and terminology).

DEFINITION 3.4. Let $\mathscr{X}$ be a collection of bounded gambles on a set of states Ω, let $\mathscr{C}$ be a set of constant gambles, and let $\gtrapprox$ be a binary relation on $\mathcal{G}[\mathscr{X}, \mathscr{C}] \coloneqq \mathscr{X} \cup \mathscr{C}$, abbreviated $\mathcal{G}$ when the context is clear. Call the pair $(\mathcal{G}, \gtrapprox)$ a *coherent comparative prevision structure* if it satisfies the following conditions:

(DF1) $(\mathcal{G}, \gtrapprox)$ is complete;

(DF2) For each gamble $f \in \mathscr{X}$, there is $c \in \mathbb{R}$ such that $\boldsymbol{c} \in \mathscr{C}$ and $f \approx \boldsymbol{c}$;

(DF3) DF COHERENCE For every $n \in \mathbb{N}_{>0}$, there are no $(f_i, g_i)_{i=1}^{n} \in \mathcal{G}^{2n}$ such that:

(DF3.1) $$f_i \gtrapprox g_i \qquad \text{for each } i = 1, \ldots, n$$

and

(DF3.2) $$\sum_{i=1}^{n} g_i \geqslant \sum_{i=1}^{n} f_i$$

yet either

(DF3.3) $$f_i \succ g_i \qquad \text{for some } i = 1, \ldots, n,$$

[8] I shall not require, however, the set of constant functions to consist solely of appraised fair prices of gambles from $\mathscr{X}$, since this is an unnecessary imposition distracting from the point.

or

$$(\text{DF3.4}) \qquad \sum_{i=1}^{n} g_i \ggg \sum_{i=1}^{n} f_i.$$

◂

Thus, the qualitative reformulation of de Finetti's coherence criterion states that there is no finite sequence of comparisons $f_1 \gtrapprox g_1, \ldots, f_n \gtrapprox g_n$ among gambles in $\mathcal{G}$ such that in combination $\sum_{i=1}^{n} g_i \geqslant \sum_{i=1}^{n} f_i$, yet either $f_i \succ g_i$ for some $i = 1, \ldots, n$ or the composite gamble $\sum_{i=1}^{n} g_i$ uniformly simply dominates the composite gamble $\sum_{i=1}^{n} f_i$. Of course, this criterion is logically equivalent to the assertion that for every finite sequence of comparisons $f_1 \gtrapprox g_1, \ldots, f_n \gtrapprox g_n$ among gambles in $\mathcal{G}$, if $\sum_{i=1}^{n} g_i \geqslant \sum_{i=1}^{n} f_i$, then $f_i \approx g_i$ for each $i = 1, \ldots, n$ and $\sum_{i=1}^{n} g_i$ does not uniformly simply dominate $\sum_{i=1}^{n} f_i$.[9]

Conditions (DF1) and (DF2) of a coherent comparative prevision structure formally articulate the assumptions that the agent is committed to being indifferent between any given gamble of interest and its fair price and that he is committed to a complete preference ranking over gambles and their fair prices. By stating these assumptions separately, de Finetti's criterion of coherence, and in particular the logically equivalent restatement of his criterion expressed in condition (DF$\mathbb{Z}$), can be reformulated in condition (DF3) to make the preferential commitments of his criterion transparent without explicitly stating the role that fair prices in $\mathscr{C}$ play as constant gambles preferentially equivalent to gambles from $\mathscr{X}$.[10] I shall discuss the interpretation of the qualitative reformulation of de Finetti's account in more detail in the next section when I compare it with my account of coherent comparative expectations.

The reader is invited to verify that conditions (DF1), (DF2), and (DF3) of a coherent comparative prevision structure $(\mathcal{G}, \gtrapprox)$ entail that (1) $\gtrapprox$ is a weak order on $\mathcal{G}$ and that (2) for each gamble $f \in \mathcal{G}$, there is a *unique* $c \in \mathbb{R}$ such that $f \approx \boldsymbol{c}$, which may be suggestively denoted by $\mathbb{P}(f)$ and its constant correlate by $\mathbf{P}(f)$. By applying Fact 2.2 and Fact 2.3, we obtain the following qualitative reformulation of de Finetti's fundamental results.

THEOREM 3.5. *Let $(\mathcal{G}[\mathscr{X}, \mathscr{C}], \gtrapprox)$ be a coherent comparative prevision structure. Then*:

(1) *There is a finitely additive expectation $\mathbb{P}$ on $\mathcal{G}$ such that for every $f, g \in \mathcal{G}$:*

$$(\text{DF FP}) \qquad f \approx \mathbf{P}(f) \quad \textit{for some} \quad \mathbf{P}(f) \in \mathscr{C},$$

$$(\text{DF REP}) \qquad f \gtrapprox g \quad \textit{if and only if} \quad \mathbb{P}(f) \geqslant \mathbb{P}(g);$$

(2) *If f is a bounded gamble on Ω, then $(\mathcal{G}, \gtrapprox)$ can be extended to a coherent comparative prevision structure $\big(\mathcal{G}[\mathscr{X} \cup \{f\}, \mathscr{C} \cup \{\widehat{\mathbf{P}}(f)\}], \gtrsim\big)$;*

(3) *If $\mathscr{Y}$ is a collection of bounded gambles on Ω such that $\mathscr{Y}$ contains $\mathscr{X}$, then $(\mathcal{G}, \gtrapprox)$ can be extended to a coherent comparative prevision structure $(\mathcal{G}[\mathscr{Y}, \mathscr{C} \cup \{\widehat{\mathbf{P}}(f) : f \in \mathscr{Y}\}], \gtrsim)$.* ■

Proof. I offer a sketch. Define $\mathbb{P} : \mathcal{G} \to \mathbb{R}$ by setting $\mathbb{P}(f)$ to be the unique real number $\mathrm{P}(f)$ such that $f \approx \mathbf{P}(f)$. Observe that $\mathbf{P}$ and $\gtrapprox$ jointly satisfy property (DF FP) and then confirm that $\mathbb{P}$ and

[9] I remark that the following results only require condition (DF2) and condition (DF3). However, to offer a faithful and charitable reformulation of de Finetti's account, I have added condition (DF1).

[10] Constant gambles in $\mathscr{C}$ that do not serve as fair prices for gambles in $\mathscr{X}$ serve as fair prices for themselves, although such gambles may as well be included in $\mathscr{X}$ for conceptual uniformity (cf. footnote 8).

$\gtrapprox$ also jointly satisfy property (DF REP). Now verify that $\mathbb{P}$ satisfies condition (DF$\mathbb{Z}$), so by Proposition 3.2 it follows that $\mathbb{P}$ satisfies de Finetti's numerical criterion of coherence. Hence, by Fact 2.2, $\mathbb{P}$ is a finitely additive expectation over $\mathcal{G}$, establishing part (1). As for part (2), apply Proposition 3.2 and part (i) of Fact 2.3 to extend $\mathbb{P}$ to a coherent prevision $\mathbb{P}_0$ on $\mathcal{G} \cup \{f\}$ and then to a coherent prevision $\widehat{\mathbb{P}}$ on $\mathcal{G} \cup \{f, \widehat{\mathbf{P}}(f)\}$, where $\widehat{\mathbf{P}}(f)$ is the constant gamble associated with $\mathbb{P}_0(f)$. Define a binary relation $\gtrsim$ on $\mathcal{G}[\mathscr{X} \cup \{f\}, \mathscr{C} \cup \{\widehat{\mathbf{P}}(f)\}]$ by setting $g \gtrsim h$ if and only if $\widehat{\mathbb{P}}(g) \geqslant \widehat{\mathbb{P}}(h)$. Verify that $\big(\mathcal{G}[\mathscr{X} \cup \{f\}, \mathscr{C} \cup \{\widehat{\mathbf{P}}(f)\}], \gtrsim\big)$ is a coherent comparative prevision structure extending $(\mathcal{G}[\mathscr{X}, \mathscr{C}], \gtrapprox)$. Property (3) similarly follows from part (ii) of Fact 2.3. □

A qualitative reformulation of Fact 2.2, part (1) says that an agent's preferences over gambles accord with a ranking by a finitely additive expectation. Part (2) is a qualitative reformulation of statement (a) of part (i) of Fact 2.3, de Finetti's Fundamental Theorem of Prevision, while part (3) is a qualitative reformulation of part (ii) of Fact 2.3, a corollary of the Fundamental Theorem of Prevision. The bounds provided in statement (b) of part (i) of Fact 2.3 remain the same. Provided that $\mathscr{X}$ is a unital $\mathbb{Q}$-linear space of $\mathbb{R}^\Omega$, if $\gtrapprox$ is a binary relation on $\mathcal{G}[\mathscr{X}, \mathscr{C}]$ satisfying part (1) of Theorem 3.5, then $(\mathcal{G}, \gtrapprox)$ is a coherent comparative prevision structure. A straightforward result, part (1) of Theorem 3.5 generalizes and extends a theorem due to Diecidue and Wakker [2002, Theorem 2]. Roughly speaking, these authors require that the collection of gambles is the set of all real-valued functions on a set of states and that the set of states is finite.

To conclude this section, I highlight an immediate consequence of the foregoing developments for comparative previsions over bounded gambles admitting fair price equivalents.

PROPOSITION 3.6. *Let $\mathscr{X}$ be a collection of bounded gambles on a set of states Ω, let $\mathscr{C}$ be a set of constant gambles, and let $\gtrapprox$ be a binary relation on $\mathcal{G}[\mathscr{X}, \mathscr{C}]$. Suppose that $(\mathcal{G}, \gtrapprox)$ satisfies properties* (DF1) *and* (DF2). *Then $(\mathcal{G}, \gtrapprox)$ satisfies property* (DF3) *if and only if it satisfies the following condition*:

(DF3$\mathbb{R}$) *For every $n \in \mathbb{N}_{>0}$, there are no $(f_i, g_i)_{i=1}^n \in \mathcal{G}^{2n}$ such that for some $(c_i)_{i=1}^n \in \mathbb{R}^n_{>0}$:*

(DF3.1$\mathbb{R}$) $$f_i \gtrapprox g_i \qquad \text{for each } i = 1, \ldots, n$$

and

(DF3.2$\mathbb{R}$) $$\sum_{i=1}^n c_i g_i \geqslant \sum_{i=1}^n c_i f_i$$

yet either

(DF3.3$\mathbb{R}$) $$f_i \succ g_i \qquad \text{for some } i = 1, \ldots, n,$$

or

(DF3.4$\mathbb{R}$) $$\sum_{i=1}^n c_i g_i \ggg \sum_{i=1}^n c_i f_i.$$

■

Informally, just as de Finetti's numerical criterion of coherence (Definition 2.1) is logically equivalent to the simple numerical criterion (DF$\mathbb{Z}$), the qualitative criterion of coherence (DF3$\mathbb{R}$) is logically equivalent to the simple qualitative criterion (DF3). Insofar as it includes real coefficients in its formulation, criterion (DF3ℝ) is closer than (DF3) is to de Finetti's original numerical criterion of coherence.

4. Coherent Comparative Expectations

The requirement that each gamble of interest has a preferentially equivalent fair price is an Archimedean condition imposed on an agent's system of preferences to ensure a real-valued representation. The viewpoint investigated in this article contends that this requirement, like other Archimedean axioms and their stand-in continuity conditions, is a technical condition unmotivated by general criteria of rational choice and indeed places substantive restrictions on what qualifies as rationality permissible systems of comparative judgments. I shall therefore drop this requirement. I shall also drop the similarly unmotivated requirement of boundedness on gambles as well as the requirement of weak ordering on comparative judgments.

While an agent may indeed adopt non-Archimedean preferences over gambles whose outcomes are aptly represented by real numerical values, an agent's rational preferential comparisons over outcomes themselves may very well form a non-Archimedean system which accordingly cannot be represented by real numerical values. For example, an agent's preferences over outcomes may exhibit a hierarchical structure that violates Archimedean conditions without conflicting with standards of rational choice (see, e.g., [Hausner, 1954], [Thrall, 1954]). With respect to the present discussion, this suggests also dropping the requirement that an agent's valuation over outcomes be representable in terms of real numbers, accordingly allowing an agent's valuation over outcomes to be representable by, for example, a non-Archimedean ordered field. To facilitate a smooth comparison with de Finetti's framework, however, I shall retain the assumption that an agent's valuation over outcomes is representable in terms of real numbers. I investigate the consequences of dropping this requirement in a general context in [Pedersen, 2013].

As I discussed in §2, de Finetti's criterion of coherence allows for violations of the principle of weak dominance. In contrast with the aforementioned requirements, I shall insist that an agent's comparative judgments respect the principle of weak dominance in accordance with the position investigated in this article.

I shall now develop aspects of an account of expectation based on a qualitative criterion of coherence for a system of comparative expectations. In this section, I shall investigate the consequences of the criterion of coherence and compare it with the qualitative reformulation of de Finetti's criterion of coherence. I shall thereupon connect the concept of a coherent system of comparative expectations, introduced next, with the concept of a coherent expectation structure, which resembles preference structures familiar from decision theory (see §1 for notational conventions and terminology).

Definition 4.1. Let $\mathscr{X}$ be a collection of gambles on a set of states Ω, let $\gtrsim$ be a binary relation on $\mathscr{X}$, and let $\mathbb{K}$ be a totally ordered subfield of $\mathbb{R}$. Call the pair $(\mathscr{X}, \gtrsim)$ a $\mathbb{K}$-*linear coherent system of comparative expectations* if it satisfies the following condition:

(C) Coherence For every $n \in \mathbb{N}_{>0}$, there are no $(f_i, g_i)_{i=1}^n \in \mathscr{X}^{2n}$ such that for some $(c_i)_{i=1}^n \in \mathbb{K}^n_{>0}$:

(C1) $$f_i \gtrsim g_i \qquad \text{for each } i = 1, \ldots, n$$

and

(C2) $$\sum_{i=1}^n c_i g_i \geqslant \sum_{i=1}^n c_i f_i$$

yet either

(C3) $$f_i \succ g_i \qquad \text{for some } i = 1, \ldots, n,$$

or

(C4) $$\sum_{i=1}^{n} c_i g_i \succ \sum_{i=1}^{n} c_i f_i.$$

◂

Thus, when $\mathbb{K} = \mathbb{R}$, a coherent system of comparative expectations must only satisfy condition (DF3$\mathbb{R}$) but with uniform simple dominance in (DF3.4$\mathbb{R}$) replaced by weak dominance in (C4). In addition, observe that when $\mathbb{K} = \mathbb{Q}$, a coherent system of comparative expectations must only satisfy condition (DF3) but again with uniform simple dominance in (DF3.4) replaced by weak dominance in (C4). Thus, in this case, the rational coefficients in (C) may be dropped. To avoid a digression on technical complications concerning scale invariance for unbounded functions (as well as questions about the normative status of scale invariance properties), condition (C) is explicitly formulated with coefficients in $\mathbb{K}$. Criterion (C) may be understood as a qualitative version of the criterion of strict coherence discussed in §2. Observe that this criterion does not entail that $\gtrsim$ is reflexive, complete, or indeed even transitive.

Without reference to fair prices, the decision-theoretic scheme rehearsed in §2 can be recast in a similar spirit for comparative expectations. Let $\mathscr{X}$ be a collection of gambles of interest to a given agent at a given time. The agent is presumed to adopt a comparative ranking $\gtrsim$ over gambles in $\mathscr{X}$ entailing, for any finite sequence of comparative judgments $f_1 \gtrsim g_1, \ldots, f_n \gtrsim g_n$ and positive $\mathbb{K}$-valued units $c_1, \ldots, c_n$, a commitment to judging that (i) the composite gamble $\sum_{i=1}^{n} c_i f_i$ is acceptable for choice when his alternative option is the composite gamble $\sum_{i=1}^{n} c_i g_i$, and a commitment to the judging that (ii) the composite gamble $\sum_{i=1}^{n} c_i g_i$ is unacceptable for choice when his alternative option is the composite gamble $\sum_{i=1}^{n} c_i f_i$ if he in addition has endorsed a comparative judgment $f_i \succ g_i$ for some $i = 1, \ldots, n$.

In the present scheme, the agent's comparative ranking $\gtrsim$ is *coherent* if there is no finite sequence of comparisons $f_1 \gtrsim g_1, \ldots, f_n \gtrsim g_n$ such that for some positive $\mathbb{K}$-valued units $c_1, \ldots, c_n$, either (SCA) the agent is committed to judging that the composite gamble $\sum_{i=1}^{n} c_i f_i$ is acceptable for choice to the exclusion of the alternative composite gamble $\sum_{i=1}^{n} c_i g_i$ when in fact, by the agent's own lights, $\sum_{i=1}^{n} c_i f_i = \sum_{i=1}^{n} c_i g_i$, or (SCB) the agent is committed to judging that the composite gamble $\sum_{i=1}^{n} c_i f_i$ is acceptable for choice when, by the agent's own lights, the alternative composite gamble $\sum_{i=1}^{n} c_i g_i$ weakly dominates the composite gamble $\sum_{i=1}^{n} c_i f_i$. Informally, coherence is the requirement that an agent's comparative judgments do not entail commitments inconsistent with standards of logical consistency and rational choice.[11]

The criterion of coherence regulating a comparative prevision structure is less amenable to such a straightforwardly cogent gloss. Adapting the present scheme to the qualitative formulation of de Finetti's criterion of coherence, an agent's comparative judgments are coherent in the sense of Definition 3.4 if there is no finite sequence of comparisons $f_1 \gtrsim g_1, \ldots, f_n \gtrsim g_n$ such that for some positive reals $c_1, \ldots, c_n$, either (DFA) the agent judges that the composite gamble $\sum_{i=1}^{n} c_i f_i$ is acceptable for choice to the exclusion of the alternative composite gamble $\sum_{i=1}^{n} c_i g_i$ when in fact, by the agent's own lights, $\sum_{i=1}^{n} c_i g_i \geqslant \sum_{i=1}^{n} c_i f_i$, or (DFB) the agent judges that the composite

[11] Compare this with [de Finetti, 1931, pp. 291-292, p. 319] as well as [de Finetti, 1937, p. 103, p. 110], [de Finetti, 1974a, pp. 8-10, p. 73], and [de Finetti, 1974b, p.150]. See also [Ramsey, 1926, pp. 84-85].

gamble $\sum_{i=1}^{n} c_i f_i$ is acceptable for choice when, by the agent's own lights, the alternative composite gamble $\sum_{i=1}^{n} c_i g_i$ *uniformly simply dominates* the composite gamble $\sum_{i=1}^{n} c_i f_i$.[12] The difficulty lies with condition (DFA), for it may very well be that while $\sum_{i=1}^{n} c_i g_i \geqslant \sum_{i=1}^{n} c_i f_i$, the gamble $\sum_{i=1}^{n} c_i f_i$ is not identical with the gamble $\sum_{i=1}^{n} c_i g_i$ and yet $\sum_{i=1}^{n} c_i g_i$ does not uniformly simply dominate $\sum_{i=1}^{n} c_i f_i$ but instead only simply dominates $\sum_{i=1}^{n} c_i f_i$ or even only weakly dominates $\sum_{i=1}^{n} c_i f_i$, and in these cases criteria of rational choice straightforwardly furnish grounds for disapprobation.

Recall the prevision function $\mathbb{P}$ of Example 2.5 for the integer lottery. Suppose that an agent's preferences $\gtrapprox$ accord with the natural ordering given by $\mathbb{P}$ defined on some linear space. Although the constant gamble $\mathbf{0}$ simply dominates the gamble f defined by $f(n) = -\frac{1}{n}$ for each $n \in \mathbb{N}_{>0}$, $\mathbf{0} \gg f$, nonetheless $\mathbb{P}(f) = 0$ and so $f \approx \mathbf{0}$. Or simply consider a probability function p on an Boolean algebra that assigns zero probability to some event E judged possible, and again suppose that an agent's preferences $\gtrsim$ accord with the natural ordering given by p. Although the gamble E weakly dominates the constant gamble $\mathbf{0}$, $E > \mathbf{0}$, nonetheless $E \sim \mathbf{0}$. In both examples, an agent's comparative judgments are coherent in the sense of the last paragraph and incoherent in the sense of Definition 4.1.

I now relate the concept of a coherent system of comparative expectations with a concept that may appear more familiar.

DEFINITION 4.2. Let Ω be a set of states, let $\mathbb{K}$ be a totally ordered subfield of $\mathbb{R}$, let $\mathscr{L}$ be a $\mathbb{K}$-linear space of $\mathbb{R}^{\Omega}$, and let $\gtrapprox$ be a binary relation on $\mathscr{L}$. Call the pair $(\mathscr{L}, \gtrapprox)$ a $\mathbb{K}$-*linear comparative expectation structure* if it satisfies the following conditions for every $f, f', g, g' \in \mathscr{L}$ and $\lambda \in \mathbb{K}$:

(Q1) If $f \gtrapprox g$ and $f' \gtrapprox g'$, then $f + f' \gtrapprox g + g'$;

(Q2) If $f \gtrapprox g$ and $\lambda > 0$, then $\lambda f \gtrapprox \lambda g$;

(Q3) If $f > g$, then $g \not\gtrapprox f$. ◂

Axiom (Q1) is called *additivity* while axiom (Q2) is called *scale invariance* (or *homotheticity*). These axioms correspond to the familiar requirement of *independence*. Axiom (Q3) is a weaker qualitative version of the *principle of weak dominance* formulated to accommodate systems of comparative judgments that are incomplete. If $\gtrapprox$ is complete, then (Q3) is logically equivalent to the requirement that if $f > g$, then $f > g$. Note the conspicuous absence of an Archimedean axiom.

The next proposition expounds some relationships between comparative expectation structures and coherent systems of comparative expectations.

PROPOSITION 4.3. *Let Ω be a set of states, let $\mathscr{L}$ be a $\mathbb{K}$-linear space of $\mathbb{R}^{\Omega}$, and let $\gtrapprox$ be a binary relation on $\mathscr{L}$. Then each of the following implies its successor*:

(i) $(\mathscr{L}, \gtrapprox)$ *is a reflexive $\mathbb{K}$-linear comparative expectation structure*;

(ii) $(\mathscr{L}, \gtrapprox)$ *is a preordered $\mathbb{K}$-linear comparative expectation structure*;

(iii) $(\mathscr{L}, \gtrapprox)$ *is a preordered $\mathbb{K}$-linear coherent system of comparative expectations.*

If in addition $(\mathscr{L}, \gtrapprox)$ is complete, then (i) *to* (iii) *are pairwise equivalent.* ■

[12]Of course, the real coefficients may be dropped. In addition, recall that in the qualitative reformulation of de Finetti's account (Definition 3.4), it is presumed that the agent's comparative judgments satisfy conditions (DF1) and (DF2).

Proof. For the implication (i) ⇒ (ii), apply reflexivity and axiom (Q1) repeatedly. The implication (iii) + Completeness ⇒ (i) is straightforwardly verified using criterion (C) and completeness. As for the implication (ii) ⇒ (iii), suppose that $(\mathscr{L}, \gtrapprox)$ is a preordered $\mathbb{K}$-linear comparative expectation structure. Let $n \in \mathbb{N}_{>0}$, $(f_i, g_i)_{i=1}^n \in \mathscr{X}^{2n}$, and $(c_i)_{i=1}^n \in \mathbb{K}_{>0}^n$ be such that (C1) $f_i \gtrapprox g_i$ for each $i = 1, \ldots, n$ and (C2) $\sum_{i=1}^n c_i g_i \geqslant \sum_{i=1}^n c_i f_i$. We must show that (¬C3) $f_i \approx g_i$ for every $i = 1, \ldots, n$ and (¬C4) $\sum_{i=1}^n c_i g_i = \sum_{i=1}^n c_i f_i$.

From axioms (Q1) and (Q2) and property (C1), it follows that $\sum_{i=1}^n c_i f_i \gtrapprox \sum_{i=1}^n c_i g_i$, so by axiom (Q3) and by property (C2) we have $\sum_{i=1}^n c_i g_i = \sum_{i=1}^n c_i f_i$, establishing (¬C4). Again, by axioms (Q1) and (Q2) and property (C1), we have $\sum_{m=1, m\neq i}^n -c_i g_m \gtrapprox \sum_{m=1, m\neq i}^n -c_i f_m$ for each $i = 1, \ldots, n$, so from axioms (Q1) and (Q2) it follows that $g_i \gtrapprox f_i$ for each $i = 1, \ldots, n$, thereby establishing (¬C3). □

In the next section I shall show that any $\mathbb{K}$-linear coherent system of comparative expectations can be extended to a weakly ordered $\mathbb{K}$-linear coherent system of comparative expectations over a $\mathbb{K}$-linear space and thus to a weakly ordered $\mathbb{K}$-linear comparative expectation structure.

5. Fundamental Theorem of Comparative Expectations

As suggested by its title, the purpose of this section is to state and prove a qualitative analogue of de Finetti's Fundamental Theorem of Prevision. I begin with a simple observation.

Lemma 5.1. *Let $(\mathscr{X}, \gtrapprox)$ be a $\mathbb{K}$-linear coherent system of comparative expectations, and let $n \in \mathbb{N}_{>0}$ and $(g_i)_{i=0}^n \in \mathscr{X}^n$ be such that $g_n = g_0$. Then*:
Consistency *If $g_i \gtrapprox g_{i+1}$ for every $i < n$, then $g_i \approx g_{i+1}$ for every $i < n$.* ■

Thus, a coherent system of comparative expectations is *consistent* in the sense of Suzumura [1976], who showed by a standard application of Zorn's Lemma that a binary relation on a set is consistent just in case it can be extended to a weak order [1976, p. 387, pp. 389-390, Theorem 3]. Consistency is intermediate in logical strength between acyclicity and transitivity. Coherence also entails acylicity but does not entail transitivity. I shall show that the qualitative criterion of coherence is a necessary and sufficient condition for a binary relation on a set of gambles to be extended to a weak order.

I first establish that in a step-by-step fashion, we can eliminate incompleteness while observing strict coherence—that is, I establish the Fundamental Theorem of Comparative Expectations.

Theorem 5.2 (Fundamental Theorem of Comparative Expectations). *Let $(\mathscr{X}, \gtrapprox)$ be a $\mathbb{K}$-linear coherent system of comparative expectations, and let f and g be gambles such that $f \not\gtrapprox g$ and $g \not\gtrapprox f$. Then $(\mathscr{X}, \gtrapprox)$ can be extended to a $\mathbb{K}$-linear coherent system of comparative expectations $(\mathscr{X} \cup \{f, g\}, \gtrsim)$ such that either $f \gtrsim g$ or $g \gtrsim f$.* ■

Proof. I offer a sketch. Consider the following conditions:

(T1) There are $m \in \mathbb{N}_{\geqslant 0}$, $(f_i, g_i)_{i=1}^m \in \mathscr{X}^{2m}$, $(c_i)_{i=1}^m \in \mathbb{K}_{>0}$, and $c \in \mathbb{K}_{>0}$ such that:

- (a1) $f_i \gtrapprox g_i$ for each $i = 1, \ldots, m$;
- (b1) $cg + \sum_{i=1}^m c_i g_i \geqslant cf + \sum_{i=1}^m c_i f_i$;
- (c1) If $f_i \approx g_i$ for every $i = 1, \ldots, m$, then $cg + \sum_{i=1}^m c_i g_i > cf + \sum_{i=1}^m c_i f_i$.

(T2) There are $n \in \mathbb{N}_{\geqslant 0}$, $(f'_j, g'_j)_{j=1}^n \in \mathscr{X}^{2n}$, $(c'_j)_{j=1}^n \in \mathbb{K}_{>0}$, and $c' \in \mathbb{K}_{>0}$ such that:

- (a2) $g'_j \;\gtrapprox\; f'_j$ for each $j = 1, \ldots, n$;
- (b2) $c' f \;+\; \sum_{j=1}^n c'_j f'_j \;\geqslant\; c' g \;+\; \sum_{j=1}^n c'_j g'_j$;
- (c2) If $f'_j \approx g'_j$ for every $j = 1, \ldots, n,$ then $c' f + \sum_{j=1}^n c'_j f'_j > c' g + \sum_{j=1}^n c'_j g'_j$.

Argue that either (T1) is false or (T2) is false. Let:

$$\begin{aligned}
\mathscr{R}_1 &\coloneqq \Big\{\big\{(f,g)\big\}, \big\{(g,f)\big\}, \big\{(f,g),(g,f)\big\}\Big\};\\
\mathscr{R}_2 &\coloneqq \Big\{\big\{(f,g)\big\}, \big\{(f,g),(g,f)\big\}\Big\};\\
\mathscr{R}_3 &\coloneqq \Big\{\big\{(g,f)\big\}, \big\{(f,g),(g,f)\big\}\Big\};\\
\mathscr{R}_4 &\coloneqq \Big\{\big\{(f,g),(g,f)\big\}\Big\}.
\end{aligned}$$

Define the collection $\mathscr{R}$ by setting:

$$\mathscr{R} \quad\coloneqq\quad \begin{cases}
\mathscr{R}_1 & \text{if } (\mathscr{X} \cup \{f,g\}, \gtrapprox \cup \{(f,g)\}) \text{ is coherent and } (\mathscr{X} \cup \{f,g\}, \gtrapprox \cup \{(g,f)\}) \text{ is coherent;}\\
\mathscr{R}_2 & \text{if } (\mathscr{X} \cup \{f,g\}, \gtrapprox \cup \{(f,g)\}) \text{ is coherent and } (\mathscr{X} \cup \{f,g\}, \gtrapprox \cup \{(g,f)\}) \text{ is incoherent;}\\
\mathscr{R}_3 & \text{if } (\mathscr{X} \cup \{f,g\}, \gtrapprox \cup \{(f,g)\}) \text{ is incoherent and } (\mathscr{X} \cup \{f,g\}, \gtrapprox \cup \{(g,f)\}) \text{ is coherent;}\\
\mathscr{R}_4 & \text{otherwise.}
\end{cases}$$

Now let $\mathcal{P} \in \mathscr{R}$, and define a binary relation $\gtrsim$ on $\mathscr{X}$ by setting:

$$(\dagger) \qquad\qquad \gtrsim \quad\coloneqq\quad \begin{cases}
\gtrapprox \cup \{(f,g)\} & \text{if (T1) is false and (T2) is true;}\\
\gtrapprox \cup \{(g,f)\} & \text{if (T2) is false and (T1) is true;}\\
\gtrapprox \cup\, \mathcal{P} & \text{if (T1) is false and (T2) is false.}
\end{cases}$$

Then $(\mathscr{X} \cup \{f,g\}, \gtrsim)$ is a $\mathbb{K}$-linear coherent system of comparative expectations extending $(\mathscr{X}, \gtrapprox)$. □

In analogy with de Finetti's Fundamental Theorem of Prevision, part (i) of Fact 2.3, the piecewise definition of $\gtrsim$ given in (†) furnishes necessary and sufficient conditions for an extension of a coherent system of comparative expectations to be coherent.

On the basis of the preceding results, we are now in a position to conclude that any coherent system of comparative expectations has a weakly ordered coherent extension.

THEOREM 5.3. *Let $(\mathscr{X}, \gtrapprox)$ be a $\mathbb{K}$-linear coherent system of comparative expectations. Then $(\mathscr{X}, \gtrapprox)$ can be extended to a weakly ordered $\mathbb{K}$-linear coherent system of comparative expectations $(\mathscr{X}, \gtrsim)$.* ■

Proof. Using Zorn's Lemma, apply Lemma 5.1 and Theorem 5.2. □

The following corollary is a qualitative analogue of the corollary of de Finetti's Fundamental Theorem of Prevision stated in part (ii) of Fact 2.3.

COROLLARY 5.4. *Let $(\mathscr{X}, \gtrapprox)$ be a $\mathbb{K}$-linear coherent system of comparative expectations, and let $\mathscr{Y} \subseteq \mathbb{R}^{\Omega}$ be such that $\mathscr{Y} \supseteq \mathscr{X}$. Then $(\mathscr{X}, \gtrapprox)$ can be extended to a weakly ordered $\mathbb{K}$-linear coherent system of comparative expectations $(\mathscr{Y}, \gtrsim)$.* ∎

Although the next proposition is an immediate consequence of Theorem 5.3, it can be proved without nonconstructive principles (i.e., Zorn's Lemma), accordingly deserving a place in the exposition.

PROPOSITION 5.5. *Let $(\mathscr{X}, \gtrapprox)$ be a $\mathbb{K}$-linear coherent system of comparative expectations. Then $(\mathscr{X}, \gtrapprox)$ can be extended to a $\mathbb{K}$-linear preordered coherent system of comparative expectations $(\mathscr{X}, \gtrsim)$.* ∎

6. DISCUSSION

In the last two sections, I developed an account of comparative expectations based on a qualitative criterion of coherence, proving an analogue of de Finetti's Fundamental Theorem of Prevision, Theorem 5.2. Like de Finetti, I have offered an account that imposes no measurability, closure, or subsidiary cardinality conditions on the collection of gambles of interest to an agent. Unlike de Finetti's account, however, my account does not require the gambles of interest to the agent to be bounded. Nor does my account require every gamble of interest to the agent to have a fair price equivalent. Furthermore, the account I have offered imposes no Archimedean condition on the agent's comparative judgments. Finally, the account I have offered accommodates systems of comparative expectations that do not form a weak order.

As I mentioned both in the introduction and in the last section, in a companion article [Pedersen, 2013] I have developed a general account of comparative expectations that additionally drops the requirement that an agent's valuation over outcomes be representable in terms of real numbers, instead permitting an agent's valuation over outcomes to be representable by, for example, elements of a partially ordered (or even a preordered) Abelian group such as a non-Archimedean ordered field extension of the system real numbers (or rational numbers). I remarked that abandoning this requirement opens the door to non-Archimedean preferences over outcomes. In the general framework of the companion article, I have also proven a representation theorem for comparative expectations. I shall now state a corollary of this representation theorem formulated in the analytic framework of the present article.

Recall that a totally ordered field $\mathbb{F}$ is *Archimedean* if for every nonzero number r of $\mathbb{F}$, there is a positive integer n such that $|r| \geqslant \frac{1}{n}$ (a positive integer n is identified with the n-fold sum $1 + \cdots + 1$). Equivalently, since a field is closed under multiplicative inverses, a totally ordered field $\mathbb{F}$ is Archimedean if for every nonzero number r in $\mathbb{F}$, there is a positive integer n such that $|r| \leqslant n$. A totally ordered field is said to be *non-Archimedean* if it is not Archimedean. A non-Archimedean field includes nonzero numbers ϵ such that $|\epsilon| < \frac{1}{n}$ for all positive integers n, called *infinitesimal numbers*. Again, since a field is closed under multiplicative inverses, a non-Archimedean totally ordered field also includes numbers H such that $|H| > n$ for all positive integers n, called *unlimited numbers*.

Every Archimedean totally ordered field is isomorphic to a subfield of the field of real numbers. Non-Archimedean totally ordered fields are more abundant. The smallest non-Archimedean totally ordered field extending $\mathbb{R}$ is the field $\mathbb{R}(\epsilon)$ of rational functions in ϵ, functions of the form $\frac{p(\epsilon)}{q(\epsilon)}$, where $p(\epsilon)$ and $q(\epsilon)$ are real polynomials such that $q(\epsilon)$ is nonzero, with a unique total order satisfying the requirement that ϵ is a positive infinitesimal. Other non-Archimedean totally ordered fields include, for example, the field of Formal Laurent Series and the field of Levi-Civita Series. The most well-known non-Archimedean ordered fields are the fields of hyperreal numbers. Keisler [1963] has shown that any hyperreal field can be obtained as a limit ultrapower of the field of real numbers.

Formulated for the special case of the analytic framework of this article, I now introduce the concept of a finitely additive expectation that shall momentarily appear in the corollary of the results from [Pedersen, 2013].

DEFINITION 6.1. Let Ω be a set of states, and let $\mathscr{L}$ be a unital $\mathbb{K}$-linear space of $\mathbb{R}^{\Omega}$. A $\mathbb{K}$-*linear finitely additive expectation* over $\mathscr{L}$ is a function $\mathbb{E} : \mathscr{L} \to \mathbb{F}$ such that $\mathbb{F}$ is a totally ordered field extension of $\mathbb{R}$ and for every $f, g \in \mathscr{L}$ and $r, s \in \mathbb{K}$:

(E1) $\mathbb{E}(\mathbf{1}) = 1$;
(E2) $\mathbb{E}(rf + sg) = r\mathbb{E}(f) + s\mathbb{E}(g)$;
(E3) If $f > g$, then $\mathbb{E}(f) > \mathbb{E}(g)$. ◂

Axiom (E1) is called *normalization* and axiom (E2) is called *linearity*. Axiom (E3) is aptly called the *principle of weak dominance*. In the presence of axiom (E2), axiom (E3) is logically equivalent to the following property:

($E_{>}$) If $f > \mathbf{0}$, then $\mathbb{E}(f) > 0$.

This property is called *strict positivity*. Thus, a $\mathbb{K}$-linear finitely additive expectation is a normalized, strictly positive $\mathbb{K}$-linear mapping taking values in a totally ordered field.

Corollary 5.4 asserts that every $\mathbb{K}$-linear coherent system of comparative expectations $(\mathscr{X}, \gtrapprox)$ can be extended to a weakly ordered $\mathbb{K}$-linear coherent system of expectations $(\mathscr{Y}, \gtrsim)$ whenever $\mathscr{X} \subseteq \mathscr{Y}$ and so in particular when $\mathscr{Y}$ is the $\mathbb{K}$-linear span of $\mathscr{X} \cup \{\mathbf{1}\}$. The following corollary of the results in the companion article asserts that any weakly ordered coherent system of comparative expectations $(\mathscr{L}, \gtrsim)$ for a unital linear space $\mathscr{L}$ of real-valued functions can be represented by an expectation function.

THEOREM 6.2. *Let Ω be a set of states, let $\mathscr{L}$ be a unital $\mathbb{K}$-linear space of $\mathbb{R}^{\Omega}$, and let $\gtrsim$ be a binary relation on $\mathscr{L}$. Then the following are equivalent*:

I. *$(\mathscr{L}, \gtrsim)$ is a weakly ordered $\mathbb{K}$-linear coherent system of comparative expectations*;
II. *$(\mathscr{L}, \gtrsim)$ is a weakly ordered $\mathbb{K}$-linear comparative expectation structure*;
III. *There is a totally ordered field extension $\mathbb{F}$ of the real numbers and a $\mathbb{K}$-linear finitely additive expectation $\mathbb{E} : \mathscr{L} \to \mathbb{F}$ such that for every $f, g \in \mathscr{L}$:*

(R) $$f \gtrsim g \quad \textit{if and only if} \quad \mathbb{E}(f) \geqslant \mathbb{E}(g).$$ ■

So a system of comparative expectations $(\mathscr{L}, \gtrsim)$ is coherent just in case $\gtrsim$ can be represented by an expectation function in the sense of (R). Taking $\mathscr{L}$ to be the unital linear space generated by the linear span of a collection of (indicator functions of) events, this result delivers a numerical probability representation of comparative probability judgments as a special case. The implications **I** ⇒ **II** and **II** ⇒ **I** follow from Proposition 4.3, and the implication **III** ⇒ **II** is easily verified.

The formal developments of the companion article deliver the implication **II** ⇒ **III**. In that article I formulate a more general concept of an expectation function, and with respect to generalizations of the concepts of a system of comparative expectations and a comparative expectation structure, I extend and employ a version of a classic result called the *Hahn Embedding Theorem* to construct the desired totally ordered field for the range of an expectation function. In the present context, the developments of the companion article entail that the expectation function $\mathbb{E}$ takes values in a totally ordered field, called a *Hahn field*, in which every number can be written as *formal power series* in a *single* infinitesimal ϵ:

$$\sum_{a \in \mathbb{G}} r_a \epsilon^a$$

where $r_a \in \mathbb{R}$ and the $a \in \mathbb{G}$ such r_a is nonzero form a well-ordered subset of a given totally ordered Abelian group $\mathbb{G}$. As such, the values under the expectation function take a very simple numerical form in terms of power series with addition and multiplication naturally defined by means of the familiar operations of addition and multiplication of power series and with a total ordering such that a non-zero power series is positive just in case the coefficient of the least exponent with a non-zero coefficient is positive.

Notably, the particular numerical representation in terms of power series provides a relief map for distinctive features of a given agent's comparative judgments. To explain, I introduce a final bit of terminology. Given elements $f, g \in \mathscr{L}$ that are not $\sim$-equivalent to $\mathbf{0}$ (i.e., such that $f \nsim \mathbf{0} \nsim g$), say that f is *infinitesimal* relative to g and that g is *infinitely large* relative to f if $|g| \succ n|f|$ for every $n \in \mathbb{N}$ (where here $|h| = \max_{\precsim}(h, -h)$). Say that elements f and g of $\mathscr{L}$ which are not $\sim$-equivalent to $\mathbf{0}$ are *Archimedean equivalent* if f is not infinitesimal relative to g and g is not infinitesimal relative to f. The family of sets of Archimedean equivalent elements of $\mathscr{L}$ partitions the set of elements of $\mathscr{L}$ which are not $\sim$-equivalent to zero into *Archimedean equivalence classes*.

Now to elaborate, in the numerical representation in terms of power series, the image of $(\mathscr{L}, \succsim)$ under $\mathbb{E}$, $\mathbb{E}(\mathscr{L})$, forms a totally ordered $\mathbb{K}$-vector space such that the ordered subcollection $\{\epsilon^a \in \mathbb{E}(\mathscr{L}) : a \in \mathbb{G}\}$ of powers of the single infinitesimal ϵ which belong to $\mathbb{E}(\mathscr{L})$ is order-isomorphic to the ordered collection of Archimedean equivalence classes of gambles from $(\mathscr{L}, \succsim)$. So in particular, if f, g in $\mathscr{L}$ are such that $f, g \succ \mathbf{0}$, then if f is infinitesimal relative to g, thus belonging to distinct Archimedean equivalence classes, then there are unique elements a_f and a_g in $\mathbb{G}$ such that a_f is greater than a_g and so ϵ^{a_f} is less than ϵ^{a_g}. Thus, the power series representation *traces* infinitely large numerical differences in $\mathbb{E}(\mathscr{L})$ to infinitely large qualitative differences in $\mathscr{L}$, usefully locating the origins of numbers of the representation. Power series representations enjoy a sharp-cut, economical character that other ordered field representations, and particular hyperreal representations, generally do not enjoy.

The companion article also includes results concerning properties of the representing totally ordered field. The interested reader is personally invited to consult the companion article for these and other results.

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