MEASUREMENT THEORY

LOGIC, PROBABILITY, AND GAMES

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1. Background

Given a nonnegative integer $n \in \mathbb{N}$, let n denote the set of positive integers that do not exceed n — that is, let $\mathrm{n} := \left\{k \in \mathbb{N}_{>0} : k \leq n\right\}$. Now let $\mathbb{X}$ be a nonempty set, and let n be a nonnegative integer. An *n-ary relation* on $\mathbb{X}$ is a subset of the *n-fold Cartesian power* $\mathbb{X}^{\mathrm{n}}$ *of* $\mathbb{X}$, the set of all functions with domain n and range $\mathbb{X}$. A member of $\mathbb{X}^{\mathrm{n}}$ is called an *n-tuple* with *components* in $\mathbb{X}$. The value an n-tuple assigns to a given integer k in its domain n is called its k^{th} *component*. As usual and whenever convenient, the k^{th} component of an n-tuple x is denoted by x_k rather than by $\mathrm{x}(k)$, while more generally an n-tuple x is denoted by $\left(x_1, \ldots, x_n\right)$ rather than by x. Since a 0-tuple with components in $\mathbb{X}$ is a function with domain $\mathbf{0}$ and range $\mathbb{X}$, the 0-fold Cartesian power $\mathbb{X}^0$ has precisely one element, the null set $\emptyset$, whereby it follows that the only 0-ary relations on $\mathbb{X}$ are the empty set $\emptyset$ and its successor $\{\emptyset\}$.

An *n-ary operation* on $\mathbb{X}$ is a function from the n-fold Cartesian power $\mathbb{X}^{\mathrm{n}}$ into $\mathbb{X}$. As usual, an n-ary operation $\circ : \mathbb{X}^{\mathrm{n}} \longrightarrow \mathbb{X}$ is identified with its *graph* $\textsc{Gr}(\circ)$, an $(n+1)$-ary relation on $\mathbb{X}$:

$$\textsc{Gr}(\circ) \quad := \quad \left\{\mathrm{x} \in \mathbb{X}^{\mathrm{n}+1} : \circ\left(x_1, \ldots, x_n\right) = x_{n+1}\right\}.$$

Observe that the domain of any 0-ary operation on $\mathbb{X}$ is the unit set $\{\emptyset\}$ and so associates the null set with precisely one element of $\mathbb{X}$. Thus each 0-ary operation on $\mathbb{X}$ corresponds to a unique element of $\mathbb{X}$, and vice versa. It follows that the graph of a 0-ary operation consists of a one-tuple whose first (and last) component is the unique element of $\mathbb{X}$ corresponding to the operation in question.

Where $\mathscr{P}$ denotes the usual power set operation, let $\mathrm{Rel}(\mathbb{X})$ denote the collection of all relations on $\mathbb{X}$ of *finite arity*:

$$\mathrm{Rel}\left(\mathbb{X}\right) \quad := \quad \bigcup_{n=0}^{\infty} \mathscr{P}\left(\mathbb{X}^{\mathrm{n}}\right).$$

A set in any given $\mathscr{P}(\mathbb{X}^{\mathrm{n}})$ is therefore an n-ary relation on $\mathbb{X}$. Finally, define a *type* to be any function $\tau : \Lambda \longrightarrow \mathbb{N}$ on some set Λ. In what follows, the domain, image, and range of a given

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mapping ψ shall be respectively denoted by $\mathrm{Dom}(\psi)$, $\mathrm{Im}(\psi)$, and $\mathrm{Ran}(\psi)$. Of course, when a set C has been identified to be a subset of the domain (range) of a mapping ψ, the image (preimage) of C under ψ shall be denoted by $\psi(\mathrm{C})$ $\big(\psi^{-1}(\mathrm{C})\big)$ as befits the occassion.

2. Model-Theoretic Foundations of Representational Measurement

Definition 2.1 (Relational Structures). Let τ be a type. A *relational structure* of *type* τ is a pair $\mathfrak{X} = \big(\mathbb{X}, \mathcal{R}\big)$ equipped with a nonempty set $\mathbb{X}$ and a function $\mathcal{R} : \mathrm{Dom}(\tau) \longrightarrow \mathrm{Rel}(\mathbb{X})$ assigning each member λ of $\mathrm{Dom}(\tau)$ to a $\tau(\lambda)$-ary relation $\mathcal{R}(\lambda)$ on $\mathbb{X}$. ◀

The first component $\mathbb{X}$ of a relational structure $\mathfrak{X} = \big(\mathfrak{X}, \mathcal{R}\big)$ is called the *carrier* of $\mathfrak{X}$. The cardinality of the carrier is called the *order* of $\mathfrak{X}$. Each member of the indexed family $\big\langle \mathcal{R}(\lambda) : \lambda \in \mathrm{Dom}(\tau) \big\rangle$ of finitary relations on $\mathbb{X}$ is called a *primitive relation* of $\mathfrak{X}$.

Definition 2.2 (Substructures). Let $\mathfrak{X} = \big(\mathbb{X}, \mathcal{R}\big)$ be a relational structure of type τ.

D2.2.1 The *restriction* $\mathfrak{X}$ to a nonempty subset $\mathbb{V}$ of $\mathbb{X}$ is the relational structure $\mathfrak{X}{\restriction}\mathbb{V} = \big(\mathbb{V}, \mathcal{R}{\restriction}\mathbb{V}\big)$ of type τ such that for each $\lambda \in \mathrm{Dom}(\tau)$:

$$\big(\mathcal{R}{\restriction}\mathbb{V}\big)(\lambda) \quad = \quad \mathcal{R}(\lambda) \cap \mathbb{V}^{\tau(\lambda)}.$$

D2.2.2 A relational structure $\mathfrak{M} = \big(\mathbb{M}, \mathcal{S}\big)$ of type τ is said to be a *substructure* of $\mathfrak{X}$ — whereby $\mathfrak{X}$ is said to be an *extension* of $\mathfrak{M}$ — if the carrier $\mathbb{M}$ of $\mathfrak{M}$ is a nonempty subset of the carrier $\mathbb{X}$ of $\mathfrak{X}$ and each primitive relation $\mathcal{S}(\lambda)$ of $\mathfrak{M}$ is the restriction of the corresponding primitive relation $\mathcal{R}(\lambda)$ of $\mathfrak{X}$ to the set $\mathbb{M}$ — that is, if $\mathfrak{M}$ is the restriction $\mathfrak{X}{\restriction}\mathbb{M}$ of $\mathfrak{X}$ to $\mathbb{M}$. ◀

Definition 2.3 (Homorphisms). Let $\mathfrak{X} = \big(\mathbb{X}, \mathcal{R}\big)$ and $\mathfrak{M} = \big(\mathbb{M}, \mathcal{S}\big)$ be relational structures of type τ. A *homomorphism* of $\mathfrak{X}$ *into* $\mathfrak{M}$ is a function $\phi : \mathbb{X} \longrightarrow \mathbb{M}$ such that for each index $\lambda \in \mathrm{Dom}(\tau)$ and $\tau(\lambda)$-tuple $\big(x_1, \ldots, x_{\tau(\lambda)}\big) \in \mathbb{X}^{\tau(\lambda)}$:

D2.3 $$\big(x_1, \ldots, x_{\tau(\lambda)}\big) \in \mathcal{R}(\lambda) \quad \text{if and only if} \quad \big(\phi(x_1), \ldots, \phi(x_{\tau(\lambda)})\big) \in \mathcal{S}(\lambda).$$

The homomorphism ϕ is thereby said to provide a *representation* of $\mathfrak{X}$ in $\mathfrak{M}$ for type τ. The set of all homomorphisms from $\mathfrak{X}$ into $\mathfrak{M}$ shall be denoted by $\mathrm{Hom}(\mathfrak{X}, \mathfrak{M})$. ◀

Remark 2.4. The notion of homomorphism from Definition 2.3 is logically stronger than its eponymous model-theoretic counterpart, which only requires the 'only if' direction of criterion D2.3 (see, for example, [Chang and Keisler, 1990], [Hodges, 1993], [Marker, 2002]). The weaker criterion requires a given mapping to *preserve* the structural properties of the initial relational structure, while the measurement-theoretic notion requires a given mapping to *represent* the structural properties of the initial relation structure. A mapping satisfying criterion D2.3 has been called a *strong homomorphism* in model-theoretic developments (see, for example, [Bell and Slomson, 1969], [Mal'cev, 1973], [Manzano, 1999]; in addition, see Chang and Keisler [1990, p. 321-322] for a closely related notion with the same name).

Definition 2.5 (Morphisms). Let $\mathfrak{X} = \big(\mathbb{X}, \mathcal{R}\big)$ and $\mathfrak{M} = \big(\mathbb{M}, \mathcal{S}\big)$ be relational structures of type τ, and let $\phi : \mathbb{X} \longrightarrow \mathbb{M}$ be a homomorphism of $\mathfrak{X}$ into $\mathfrak{M}$.

D2.5.1 The mapping ϕ is called an *embedding* of $\mathfrak{X}$ *into* $\mathfrak{M}$ if ϕ is injective;

D2.5.2 The mapping ϕ is called an *epimorphism* of $\mathfrak{X}$ *onto* $\mathfrak{M}$ if ϕ is surjective;

D2.5.3 The mapping ϕ is called an *isomorphism* of $\mathfrak{X}$ *onto* $\mathfrak{M}$ if ϕ is bijective;

D2.5.4 The mapping ϕ is called an *endomorphism* of $\mathfrak{X}$ if $\mathbb{X} = \mathbb{M}$;

D2.5.5 The mapping ϕ is called an *automorphism* of $\mathfrak{X}$ if $\mathbb{X} = \mathbb{M}$ and ϕ is bijective.

◀

Observe that a function $\phi : \mathbb{X} \longrightarrow \mathbb{M}$ is a homomorphism of $\mathfrak{X} = \big(\mathbb{X}, \mathcal{R}\big)$ into $\mathfrak{M} = \big(\mathbb{M}, \mathcal{S}\big)$ if and only if the truncated mapping $\boldsymbol{\phi} : \mathbb{X} \longrightarrow \phi(\mathbb{M})$ such that $\mathrm{Gr}(\boldsymbol{\phi}) = \mathrm{Gr}(\phi)$ is an epimorphism of $\mathfrak{X}$ onto the restriction $\mathfrak{M}{\restriction}\phi(\mathbb{X})$ of $\mathfrak{M}$ to $\phi(\mathbb{X})$. Also observe that $\mathfrak{X}$ is a substructure of $\mathfrak{M}$ if and only if $\mathbb{X} \subseteq \mathbb{M}$ and the inclusion map $\iota : \mathbb{X} \hookrightarrow \mathbb{M}$ is an embedding of $\mathfrak{X}$ into $\mathfrak{M}$. It follows that a function $\phi : \mathbb{X} \longrightarrow \mathbb{M}$ is a embedding of $\mathfrak{X} = \big(\mathbb{X}, \mathcal{R}\big)$ into $\mathfrak{M} = \big(\mathbb{M}, \mathcal{S}\big)$ if and only if the truncated mapping $\boldsymbol{\phi} : \mathbb{X} \longrightarrow \phi(\mathbb{M})$ such that $\mathrm{Gr}(\boldsymbol{\phi}) = \mathrm{Gr}(\phi)$ is an isomorphism of $\mathfrak{X} = \big(\mathbb{X}, \mathcal{R}\big)$ onto $\mathfrak{M}{\restriction}\phi(\mathbb{X}) = \big(\phi(\mathbb{X}), \mathcal{S}{\restriction}\phi(\mathbb{X})\big)$ of $\mathfrak{M}$ to $\phi(\mathbb{X})$.

Definition 2.6 (Representation Profiles and Schemes). Let τ be a type. A *representation profile* of type τ is a triple $\mathsf{P} = \big(\mathfrak{X}, \mathfrak{M}, \mathcal{S}\big)$ for which $\mathfrak{X}$ and $\mathfrak{M}$ are relational structures of type τ and $\mathcal{S}$ is a subfamily of the family $\mathsf{Hom}(\mathfrak{X}, \mathfrak{M})$ called a *representation scheme* for $\mathfrak{X}$ in $\mathfrak{M}$. The *image* $\mathrm{Im}(\mathsf{P})$ of the representation profile P is the set-theoretic union of the family of all images of homomorphisms from $\mathcal{S}$:

$$\mathrm{Im}\big(\mathsf{P}\big) \quad = \quad \bigcup_{\phi \in \mathcal{S}} \mathrm{Im}(\phi)$$

A representation profile $\mathsf{P} = \big(\mathfrak{X}, \mathfrak{M}, \mathcal{S}\big)$ is said to be (*real*) *scalar* if the carrier of $\mathfrak{M}$ is a subset of the set $\mathbb{R}$ of real numbers, in which case the representation scheme $\mathcal{S}$ is called a (*real-valued*) *scale*. ◀

Definition 2.7 (Admissible Transformations). Let $\mathsf{P} = \big(\mathfrak{X}, \mathfrak{M}, \mathcal{S}\big)$ be a representation profile of type τ, and let $\phi : \mathbb{X} \longrightarrow \mathbb{M}$ be a homomorphism belonging to the representation scheme $\mathcal{S}$ of P. An *admissible transformation* of P is a function $\mathsf{T} : \mathrm{Im}(\mathsf{P}) \longrightarrow \mathbb{M}$ such that functional composition $\mathsf{T} \circ \phi : \mathbb{X} \longrightarrow \mathbb{M}$ of T with any homomorphism $\phi : \mathbb{X} \longrightarrow \mathbb{M}$ belonging to $\mathcal{S}$ is also a member of $\mathcal{S}$. The family of all admissible transformations of P shall be denoted by $\mathrm{Tr}(\mathsf{P})$. The representation profile P is said to be *regular* if for every $\phi : \mathbb{X} \longrightarrow \mathbb{M}$ and $\psi : \mathbb{X} \longrightarrow \mathbb{M}$ belonging to $\mathcal{S}$, there is an admissible transformation $\mathsf{T} : \mathrm{Im}(\mathsf{P}) \longrightarrow \mathbb{M}$ in $\mathrm{Tr}(\mathsf{P})$ such that $\mathsf{T} \circ \phi = \psi$. ◀

Figure 1 gives examples of common varieties of admissible transformations for regular scalar representation profiles.

Admissible Transformations	Scale Family	Examples
$\mathsf{T} = \mathrm{id}$ Identity Function	Absolute	Counting Probability
$\mathsf{T} = \mathbf{c}\,\mathrm{id}, \quad \mathbf{c} > 0$ Similarity Transformation	Ratio	Mass Length
$\mathsf{T} = \mathbf{c}\,\mathrm{id} + \mathbf{d}, \quad \mathbf{c} > 0$ Positive Linear Transformation	Interval	Temperature Cardinal Utility
$\mathsf{T} = \max\Big(\mathsf{T}, \sup_{\epsilon \downarrow 0} \mathsf{T} \circ (\mathrm{id} - \epsilon)\Big)$ Strictly Increasing Transformation	Ordinal	Hardness Academic Grades
$\mathsf{T} = (\mathsf{T}^{-1})^{-1}$ One-to-One Transformation	Nominal	Religion Treatment Group

Figure 1. Common Admissible Transformations

3. Extensive Measurement

Definition 3.1 (Closed Extensive Structures). Let $\mathbb{L}$ be a nonempty set, let $\succsim$ be a binary relation on $\mathbb{L}$, and let $\oplus$ be a binary operation on $\mathbb{L}$. The triple $\mathfrak{L} = \big(\mathbb{L}, \succsim, \oplus\big)$ is called a *closed extensive structure* if it for every $\mathfrak{s}, \mathfrak{t}, \mathfrak{u} \in \mathbb{L}$:

D3.1.0 Non-Triviality $\big(\mathfrak{z} \oplus \mathfrak{z}\big) \succ \mathfrak{z}$ for some $\mathfrak{z} \in \mathbb{L}$;

D3.1.1 Completeness Either $\mathfrak{s} \succsim \mathfrak{t}$ or $\mathfrak{t} \succsim \mathfrak{s}$;

D3.1.2 Transitivity If $\mathfrak{s} \succsim \mathfrak{t}$ and $\mathfrak{t} \succsim \mathfrak{u}$, then $\mathfrak{s} \succsim \mathfrak{u}$;

D3.1.3 Associativity $\big(\mathfrak{s} \oplus \mathfrak{t}\big) \oplus \mathfrak{u} \sim \mathfrak{s} \oplus \big(\mathfrak{t} \oplus \mathfrak{u}\big)$;

D3.1.4 Commutativity $\big(\mathfrak{s} \oplus \mathfrak{t}\big) \sim \big(\mathfrak{t} \oplus \mathfrak{s}\big)$;

D3.1.5 1-Monotonicity $\mathfrak{s} \succsim \mathfrak{t}$ if and only if $\big(\mathfrak{s} \oplus \mathfrak{u}\big) \succsim \big(\mathfrak{t} \oplus \mathfrak{u}\big)$;

D3.1.6 2-Monotonicity $\mathfrak{s} \succsim \mathfrak{t}$ if and only if $\big(\mathfrak{u} \oplus \mathfrak{s}\big) \succsim \big(\mathfrak{u} \oplus \mathfrak{t}\big)$.

A closed extensive structure $\mathfrak{L} = \big(\mathbb{L}, \succsim, \oplus\big)$ is said to be *positive* if for every $\mathfrak{s}, \mathfrak{t} \in \mathbb{L}$:

D3.1.7 Positivity $\big(\mathfrak{s} \oplus \mathfrak{t}\big) \succ \mathfrak{s}$ and $\big(\mathfrak{s} \oplus \mathfrak{t}\big) \succ \mathfrak{t}$.

Let $\left(\cdot,\cdot\right)$ denote the function on $\mathbb{N}_{>0}\times\mathfrak{D}$ such that for each $\mathfrak{s}\in\mathfrak{D}$ and positive integer $n\in\mathbb{N}_{>0}$:

$$\left(n,\mathfrak{s}\right) \quad = \quad \begin{cases} \mathfrak{s} & \text{if } \ n = 1; \\ \left(\left(n-1,\mathfrak{s}\right) \ + \ \mathfrak{s}\right) & \text{otherwise.} \end{cases}$$

A closed extensive structure $\mathfrak{L} = \left(\mathbb{L}, \succsim, \oplus\right)$ is said to be *Archimedean* if for every $\mathfrak{s},\mathfrak{t},\mathfrak{u},\mathfrak{v}\in\mathbb{L}$:

D3.1.8 Archimedes If $\mathfrak{s} \succ \mathfrak{t}$, then $\left(n,\mathfrak{s}\right)\oplus\mathfrak{u} \ \succsim \ \left(n,\mathfrak{t}\right)\oplus\mathfrak{v}$ for some $n\in\mathbb{N}_{>0}$.

Theorem 3.2 (Representability). *A triple $\mathfrak{L} = \left(\mathbb{L}, \succsim, \oplus\right)$ is an Archimedean closed extensive structure if and only if there is a homomorphism of $\mathfrak{L}$ into the totally ordered additive group $\mathfrak{R}_{\geq,+} = \left(\mathbb{R}, \geq, +\right)$ of real numbers — that is, there is a mapping $\phi:\mathfrak{L}\longrightarrow\mathbb{R}$ such that for $\mathfrak{s},\mathfrak{t}\in\mathfrak{L}$:*

D3.2.1 Additivity $\phi\left(\mathfrak{s}\oplus\mathfrak{t}\right) = \phi(\mathfrak{s}) \ + \ \phi(\mathfrak{t})$;

D3.2.2 Order $\mathfrak{s} \ \succeq \ \mathfrak{t}$ if and only if $\phi(\mathfrak{s}) \ \geq \ \phi(\mathfrak{t})$.

If in addition $\mathfrak{L}$ is positive, then $\phi(\mathfrak{s}) > 0$ for every $\mathfrak{s}\in\mathbb{L}$. ▪

Theorem 3.3 (Uniqueness). *Let $\mathfrak{L} = \left(\mathbb{L}, \succsim, \oplus\right)$ be a closed extensive structure, let $\phi:\mathfrak{L}\longrightarrow\mathbb{R}$ be a homomorphism of $\mathfrak{L}$ into the totally ordered additive group $\mathfrak{R}_{\geq,+} = \left(\mathbb{R}, \geq, +\right)$ of real numbers, and let $\psi:\mathfrak{L}\longrightarrow\mathbb{R}$ be a mapping from $\mathfrak{L}$ into $\mathbb{R}$. Then ψ is also a homomorphism of $\mathfrak{L}$ into $\mathfrak{R}_{\geq,+}$ if and only if $\phi \ = \ \mathfrak{c}\,\psi$ for some positive real number $\mathfrak{c}\in\mathbb{R}_{>0}$.* ▪

We will see how these fundamental results from measurement theory follow from Hölder's results, which we turn to next.

4. Hölder's Theorems

Definition 4.1. A closed extensive structure $\mathfrak{G} = \left(\mathbb{G}, \succsim, \oplus\right)$ is called a *weakly ordered Abelian group* if for every $\mathfrak{s},\mathfrak{t}\in\mathbb{G}$:

D4.1 Inversion $\left(\mathfrak{s}\oplus\mathfrak{t}\right)\sim\mathfrak{e}$ for some $\mathfrak{e}\in\mathbb{G}$.

If in addition $\succsim$ is a total ordering of $\mathbb{G}$, then $\mathfrak{G}$ is said to be a *totally ordered Abelian group.* ◀

Proposition 4.2. *Every closed extensive structure $\mathfrak{L} = \left(\mathbb{L}, \succsim, \oplus\right)$ can be embedded into a weakly ordered Abelian group $\mathfrak{G} = \left(\mathbb{G}, \precapprox, \boxplus\right)$ such that if $\mathfrak{L}$ is Archimedean, then so is $\mathfrak{G}$. Moreover, if $\left(\mathbb{L}, \succsim\right)$ is a total order, then $\mathfrak{L}$ can be embedded into a totally ordered Abelian group $\mathfrak{G} = \left(\mathbb{G}, \geq, \dotplus\right)$.*

Proof. *Left to Reader*. *Hint*: Consider the set $\mathbb{G} := \mathbb{L}\times\mathbb{L}$ and the binary operation $\boxplus$ on $\mathbb{G}$ for which $(\mathfrak{s},\mathfrak{t})\boxplus(\mathfrak{u},\mathfrak{v}) := (\mathfrak{s}\oplus\mathfrak{u},\mathfrak{t}\oplus\mathfrak{v})$ for all $\mathfrak{s},\mathfrak{t},\mathfrak{u},\mathfrak{v}\in\mathbb{L}$. Is this well-defined? What binary relation $\precapprox$ on $\mathbb{G}$ might result in a weakly ordered Abelian group? Is *it* well-defined? What mapping from $\mathbb{L}$ into the resulting structure might be the desired monomorphism? Is it well-defined? Why must this construction be modified to secure a total ordered Abelian group? □

Proposition 4.3. *Suppose that $\mathfrak{G} = \left(\mathbb{G}, \succsim, \oplus \right)$ is a weakly ordered Abelian group. Then there is a homomorphism of $\mathfrak{G}$ into a totally ordered Abelian group $\mathfrak{S} = \left(\mathbb{S}, \geq, \dotplus \right)$ such that if $\mathfrak{G}$ is Archimedean, then so is $\mathfrak{S}$.* ▪

Proof. *Left to Reader*. *Hint*: Adapt the proof of Proposition 4.2.

□

Theorem 4.4 (1901, Hölder's Theorem I.a). *Every Archimedean totally ordered Abelian group $\mathfrak{G} = \left(\mathbb{G}, \geq, \dotplus \right)$ can be embedded into the totally ordered additive group $\mathfrak{R}_{\geq,+} = \left(\mathbb{R}, \geq, + \right)$ of real numbers.*

Proof Sketch. Let $\mathfrak{u} \in \mathbb{G}_{>\mathfrak{e}}$ be a positive element of $\mathfrak{G}$ — thus $\mathfrak{u} > \mathfrak{e}$, where $\mathfrak{e}$ denotes the unique additive identity of $\mathfrak{G}$. For each $\mathfrak{s} \in \mathbb{G}_{\geq\mathfrak{e}}$, let $\mathcal{L}(\mathfrak{s})$ and $\mathcal{U}(\mathfrak{s})$ denote the following sets:

T4.4.1 $$\mathcal{L}(\mathfrak{s}) := \left\{ \frac{m}{n} \in \mathbb{Q} : \left(n, \mathfrak{s}\right) \geq \left(m, \mathfrak{u}\right) \text{ for some } m, n \in \mathbb{N}_{>0} \right\};$$

T4.4.2 $$\mathcal{U}(\mathfrak{s}) := \left\{ \frac{m}{n} \in \mathbb{Q} : \left(m, \mathfrak{u}\right) \geq \left(n, \mathfrak{s}\right) \text{ for some } m, n \in \mathbb{N}_{>0} \right\}.$$

Argue that $\mathcal{L}(\mathfrak{s})$ is a nonempty subset of $\mathcal{U}(\mathfrak{s})$ for every positive element $\mathfrak{s} \in \mathbb{G}_{>\mathfrak{e}}$.

Now let $\phi_{\geq} : \mathbb{G}_{\geq\mathfrak{e}} \longrightarrow \mathbb{R}_{\geq 0}$ be the mapping defined by setting for each $\mathfrak{s} \in \mathbb{G}_{\geq\mathfrak{e}}$:

$$\phi_{\geq}(\mathfrak{s}) := \inf \mathcal{U}(\mathfrak{s}).$$

Verify that $\phi_{\geq}$ is a well-defined mapping such that for each positive element $\mathfrak{s} \in \mathbb{G}_{>\mathfrak{e}}$:

$$\phi_{\geq}(\mathfrak{s}) = \sup \mathcal{L}(\mathfrak{s}).$$

Establish that $\phi_{\geq}$ satisfies the following properties for every $\mathfrak{s}, \mathfrak{t} \in \mathbb{G}_{\geq\mathfrak{e}}$:

$$\phi_{\geq}\left(\mathfrak{u}\right) = 1;$$

$$\phi_{\geq}\left(\mathfrak{e}\right) = 0;$$

$$\phi_{\geq}(\mathfrak{s} \dotplus \mathfrak{t}) = \phi_{\geq}(\mathfrak{s}) + \phi_{\geq}(\mathfrak{t});$$

$$\mathfrak{s} \geq \mathfrak{t} \quad \text{if and only if} \quad \phi_{\geq}(\mathfrak{s}) \geq \phi_{\geq}(\mathfrak{t}).$$

Now extend $\phi_{\geq}$ to all of $\mathbb{G}$ by defining $\phi : \mathbb{G} \longrightarrow \mathbb{R}$ to be such that for each $\mathfrak{s} \in \mathbb{G}$:

$$\phi(\mathfrak{s}) := \begin{cases} \phi_{\geq}(\mathfrak{s}) & \text{if } \mathfrak{s} \in \mathbb{G}_{\geq\mathfrak{e}}; \\ -\phi_{\geq}(\mathfrak{s}) & \text{otherwise.} \end{cases}$$

Verify that ϕ is well-defined. Finally, confirm that ϕ is a homomorphism of $\mathfrak{G}$ into $\mathfrak{R}_{\geq,+}$, thereby concluding the proof. □

Theorem 4.5 (Hölder's Theorem I.b). *Let $\phi : \mathbb{G} \longrightarrow \mathbb{R}$ be an embedding of a totally ordered group $\mathfrak{G} = \left(\mathbb{G}, \geq, \dotplus \right)$ into the totally ordered additive group $\mathfrak{R}_{\geq,+} = \left(\mathbb{R}, \geq, + \right)$ of the field of real numbers, and let $\psi : \mathbb{G} \longrightarrow \mathbb{R}$ be a mapping from $\mathbb{G}$ into $\mathbb{R}$. Then ψ is also a embedding of $\mathfrak{G}$ into $\mathfrak{R}_{\geq,+}$ if and only if $\phi = c\psi$ for some positive real number $c \in \mathbb{R}_{>0}$.*

Remark 4.6. Theorem 4.4 remains valid without the separate assumption that $\mathfrak{G} = \left(\mathbb{G}, \geq, \dot{+}\right)$ is Abelian — that is, if property D3.1.4, Commutativity, is dropped. Although Hölder originally formulated his result without this separate assumption, a direct argument establishes that every Archimedean totally order group is Abelian (see, for example, [Fuchs, 1963], [Pfanzagl, 1968], [Krantz et al., 1971], [Narens, 1985]; this is a corollary of Hölder's theorem, of course). It is not true, however, that every totally ordered Abelian group is Archimedean (*why?*). Retracing our path, it follows that in the presence of properties D3.1.1, D3.1.2, D3.1.3, D3.1.5, and D3.1.6, property D3.1.4, property D3.1.8, Archimedes, is logically stronger than property D3.1.4, Commutativity.

Two straightforward applications of the preceding theorem for totally ordered groups suffice to establish Hölder's theorem for Archimedean totally ordered fields.

Theorem 4.7 (Hölder's Theorem II). *Every Archimedean totally ordered field* $\mathfrak{F} = \left(\mathbb{F}, \geq, \dot{+}, \dot{\times}\right)$ *can be embedded into the totally ordered field* $\mathfrak{R}_{\geq,+,\times} = \left(\mathbb{R}, \geq, +, \times\right)$ *of real numbers.* ▪

Proof. *Left to Reader*. *Hint*: Apply the technique outlined in the proof of Theorem 4.4.

□

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